A FUBINI THEOREM

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Abstract. Let $\mathcal{I}_0$ be the σ-ideal of subsets of a Polish group generated by Borel sets which have perfectly many pairwise disjoint translates. We prove that a Fubini-type theorem holds between $\mathcal{I}_0$ and the σ-ideals of Haar measure zero sets and of meager sets. We use this result to give a simple proof of a generalization of a theorem of Balcerzak-Rosłanowski-Shelah stating that $\mathcal{I}_0$ on $2^{\mathbb{N}}$ strongly violates the countable chain condition.

Apart from the original Fubini theorem for measure zero sets and the Kuratowski-Ulam theorem for meager sets, theorems relating smallness, understood as membership in a σ-ideal, of a subset of a product of Polish spaces and smallness of its sections frequently fail to hold. I will show that a Fubini-type theorem does hold between σ-ideals such as the Haar measure zero ideal or the meager ideal and the σ-ideal generated by Borel subsets of a Polish group which have perfectly many pairwise disjoint translates. This last σ-ideal was introduced in [2], and here is its precise definition. Let G be a Polish group. Let $\mathcal{F}_0(G)$ stand for the family of all Borel sets B such that there exists a Cantor set $K \subseteq G$ with $xB \cap yB = \emptyset$ for $x, y \in K$ and $x \neq y$. Define $\mathcal{I}_0(G)$ to be the σ-ideal generated by $\mathcal{F}_0(G)$.

For proofs of failure of related, but different, Fubini-type theorems relating smallness of vertical sections to smallness of horizontal sections of a subset of a product of Polish spaces the reader may consult [3], [5], and [6].

1. Uniformly Steinhaus ideals. First we isolate some properties of the σ-ideal of meager sets and of Haar measure zero sets that will be important later. The first of these properties is a definability condition on the ideal, the second one is related to Steinhaus' theorem. If $A \subseteq X \times Y$ and $x \in X$, we write A_x for $\{y \in Y : (x, y) \in A\}$. Let $\mathcal{I}$ be a σ-ideal of subsets of a Polish group G. We say that it is *uniformly Steinhaus* if $\mathcal{I} = \bigcap_{n\in\mathbb{N}} \mathcal{G}_n$ and each $\mathcal{G}_n$ fulfills the following two conditions:

(i) for any Polish space X and a Borel set $B \subseteq X \times G$ the set $\{x \in X : B_x \in \mathcal{G}_n\}$ is Borel;

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(ii) there exists a neighborhood U of 1 such that for any Borel $A, B \subseteq G$, if $A, B \notin \mathcal{G}_n$, then $U \subseteq AB^{-1}$.

Lemma. *Let G be a Polish group. The σ-ideal of meager sets is uniformly Steinhaus. If G is additionally locally compact, then the σ-ideal of Haar measure zero sets is uniformly Steinhaus.*

Proof. Let $\{U_n : n \in \mathbb{N}\}$ be a basis of the topology on G consisting of nonempty open sets. If G is locally compact, we assume that $\overline{U}_n$ is compact for each n. For $\mathcal{I}$ the σ-ideal of meager sets, put

$$\mathcal{G}_n = \{A \subseteq G : U_n \setminus A \text{ is not meager}\}.$$

For $\mathcal{I}$ the σ-ideal of Haar measure zero sets (G locally compact), let

$$\mathcal{G}'_n = \{A \subseteq G : \lambda_*(U_n \setminus A) > \frac{1}{3}\lambda(U_n)\}$$

where λ is the left Haar measure and λ_* is the inner measure associated with λ. Point (i) in the definition of uniformly Steinhaus ideals follows from [4, Theorem 16.1] for $\mathcal{G}_n$ and from [4, Theorem 17.25] for $\mathcal{G}'_n$. Point (ii) is easily checked from translation invariance of the meager ideal for $\mathcal{G}_n$ and from left translation invariance of λ for $\mathcal{G}'_n$ (for a similar argument see [4, Theorem 9.9]). □

2. A Fubini theorem. We have the following Fubini-type result. (The definition of the σ-ideal $\mathcal{I}_0$ is given in the introduction.)

Theorem. *Let G_1, G_2 be Polish groups, and let the σ-ideal $\mathcal{I}$ on G_2 be uniformly Steinhaus. If $A \subseteq G_1 \times G_2$ and $A \in \mathcal{I}_0(G_1 \times G_2)$, then*

$$\{x \in G_1 : A_x \notin \mathcal{I}\} \in \mathcal{I}_0(G_1).$$

Proof. Let $\mathcal{G}_n$, $n \in \mathbb{N}$, be chosen for $\mathcal{I}$ to witness that this ideal is uniformly Steinhaus. We assume that $\{x \in G_1 : A_x \notin \mathcal{I}\} \notin \mathcal{I}_0(G_1)$ and prove that $A \notin \mathcal{I}_0(G_1 \times G_2)$. Let C_m, $m \in \mathbb{N}$, be Borel sets with $A \subseteq \bigcup_m C_m$. We need to see that for some m, $C_m \notin \mathcal{F}_0(G_1 \times G_2)$. (The definition of $\mathcal{F}_0$ is given in the introduction as part of the definition of $\mathcal{I}_0$.) Put $B = \{x \in G_1 : A_x \notin \mathcal{I}\}$. Define

$$B_{n,m} = \{x \in G_1 : (C_m)_x \notin \mathcal{G}_n\}. \tag{1}$$

By (i) of the definition of uniformly Steinhaus ideals, each $B_{n,m}$ is Borel. Since $\mathcal{I} = \bigcap_n \mathcal{G}_n$, $B \subseteq \bigcup_{n,m} B_{n,m}$. Since $B \notin \mathcal{I}_0(G_1)$, there exists n_0, m_0 such that $B_{n_0,m_0} \notin \mathcal{F}_0(G_1)$. We claim that $C_{m_0} \notin \mathcal{F}_0(G_1 \times G_2)$.

Towards a contradiction assume otherwise, and fix a Cantor set $K \subseteq G_1 \times G_2$ witnessing $C_{m_0} \in \mathcal{F}_0(G_1 \times G_2)$. Let π_1, π_2 be the projections onto G_1 and G_2, respectively. Using (ii) of the definition of uniformly Steinhaus ideals for $\mathcal{I}$, we can find a relatively clopen, nonempty subset V of K such that $\pi_2(V)^{-1}\pi_2(V) \subseteq PQ^{-1}$ for all Borel sets $P, Q \subseteq G$ with $P, Q \notin \mathcal{G}_{n_0}$. This condition translates to

$$\forall y, z \in V \forall P, Q \text{ Borel sets } (P, Q \notin \mathcal{G}_{n_0} \Rightarrow \pi_2(y)P \cap \pi_2(z)Q \neq \emptyset). \tag{2}$$

Since V is a Cantor set as well, we can assume that $V = K$. It follows from (2) and (1) that

$$\forall y, z \in K \forall w, x \in B_{n_0,m_0}\ \pi_2(y)(C_{m_0})_w \cap \pi_2(z)(C_{m_0})_x \neq \emptyset. \tag{3}$$

Now we also have that for some $x_0, y_0 \in K$ with $x_0 \neq y_0$,

$$\pi_1(x_0)B_{n_0,m_0} \cap \pi_1(y_0)B_{n_0,m_0} \neq \emptyset. \tag{4}$$

To achieve this either find $x_0, y_0 \in K$ with $x_0 \neq y_0$ and $\pi_1(x_0) = \pi_1(y_0)$ or, if $\pi_1 \restriction K$ is injective, use the fact that $B_{n_0,m_0} \notin \mathcal{F}_0(G_1)$. Pick w in the intersection in (4), and let $w_1 = (\pi_1(x_0))^{-1}w$ and $w_2 = (\pi_1(y_0))^{-1}w$. Now using (3), pick v in $\pi_2(x_0)(C_{m_0})_{w_1} \cap \pi_2(y_0)(C_{m_0})_{w_2}$. Then we have $(w_1, \pi_2(x_0)^{-1}v) \in C_{m_0}$ and $(w_2, \pi_2(y_0)^{-1}v) \in C_{m_0}$, hence, by the definition of w_1 and w_2, $(w, v) \in x_0C_{m_0}$ and $(w, v) \in y_0C_{m_0}$ contradicting our assumption that $xC_{m_0} \cap yC_{m_0} = \emptyset$ for any distinct $x, y \in K$. □

By using Galvin's theorem [4, Theorem 19.7] in the proof of Theorem, we can weaken (ii) in the definition of uniformly Steinhaus ideals to (ii') below and still have the same result.

(ii') Each Cantor set $P \subseteq G$ contains two points $y, z \in P$, $y \neq z$ such that for any Borel $A, B \notin \mathcal{G}_n$, $yA \cap zB \neq \emptyset$.

3. An application. From the theorem above we will derive a generalization of a theorem of Balcerzak, Rosłanowski, and Shelah [2]. We say that an ideal of subsets of a Polish space X has *property M* if there exists a Borel function $f : X \to 2^{\mathbb{N}}$ such that preimages of all points in $2^{\mathbb{N}}$ are not in the ideal. Some applications of this strong violation of the countable chain condition can be found in [1]. It is proved in [2, Theorem 2.1] that $\mathcal{I}_0(2^{\mathbb{N}})$ has property M, where $2^{\mathbb{N}}$ is considered with the coordinatewise addition modulo 2. We generalize this result here and also provide a simpler proof of it. To see that the corollary below is a generalization of [2, Theorem 2.1] note that $2^{\mathbb{N}}$ is a Polish uncountable compact group and that $2^{\mathbb{N}}$ is isomorphic as a topological group with $(2^{\mathbb{N}})^{\mathbb{N}}$.

Corollary. *Let each G_n, $n \in \mathbb{N}$, be a compact, Polish, uncountable group. Then $\mathcal{I}_0(\prod_n G_n)$ has property M.*

Proof. It follows from Lemma and Theorem that if H_1, H_2 are Polish compact, $A \subseteq H_1$ is not Haar measure zero and $B \subseteq H_2$ is not meager, then $A \times B \notin \mathcal{I}_0(H_1 \times H_2)$. By taking $H_1 = \{1\}$ or $H_2 = \{1\}$ we see that, in particular, non-Haar measure zero subsets of a Polish compact group H are not in $\mathcal{I}_0(H)$ and similarly for non-meager subsets.

Now, for each $n \in \mathbb{N}$, let A_n^0, A_n^1 be two disjoint subsets of G_n whose union is G_n and which are such that A_n^0 is an F_σ of full Haar measure while A_n^1 is a dense G_δ. Define $f : \prod_n G_n \to 2^{\mathbb{N}}$ by letting $f((x_n))(i) = 0$ if $x_i \in A_i^0$ and $f((x_n))(i) = 1$ if $x_i \in A_i^1$. One easily checks that f is Borel. Let now $y \in 2^{\mathbb{N}}$. Assume that y is not constantly 0 or constantly 1. Let $Y_1 = \{n \in \mathbb{N} : y(n) = 0\}$ and $Y_2 = \{n \in \mathbb{N} : y(n) = 1\}$. Then there is a canonical topological group isomorphism between $\prod_n G_n$ and $\prod_{n\in Y_1} G_n \times \prod_{n\in Y_1} G_n$. This isomorphism carries $f^{-1}(y)$ to $\prod_{n\in Y_1} A_n^0 \times \prod_{n\in Y_2} A_n^1$. Note that $\prod_{n\in Y_1} A_n^0$ has full Haar measure (which is the product measure of Haar measures on the groups G_n with $n \in Y_1$) in $\prod_{n\in Y_1} G_n$ and $\prod_{n\in Y_2} A_n^1$ is comeager in $\prod_{n\in Y_2} G_n$. Therefore, by what was said above with $H_1 = \prod_{n\in Y_1} G_n$ and $H_2 = \prod_{n\in Y_2} G_n$, $f^{-1}(y)$ is not in $\mathcal{I}_0(\prod_n G_n)$. If y is constantly 0 or constantly 1, then $f^{-1}(y)$ has full Haar measure in $\prod_n G_n$ in the first case and is comeager in the second case. In either case, as indicated at the beginning of this proof, $f^{-1}(y)$ is not in $\mathcal{I}_0(\prod_n G_n)$. □

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