A RAMSEY THEOREM FOR STRUCTURES WITH BOTH RELATIONS AND FUNCTIONS

SŁAWOMIR SOLECKI

Abstract. We prove a generalization of Prömel's theorem to finite structures with both relations and functions.

1. Introduction

The classical Ramsey theorem [11] has been generalized in two (among other) important, independent directions. First, Graham and Rothschild [3] extended Ramsey's theorem from finite sets to finite parameter sets, objects more general than sets, whose particular examples include also partitions. (Parameter sets are sometimes called combinatorial cubes.) Second, Abramson and Harrington [1] and Nešetřil and Rödl [6], [7] extended Ramsey's theorem to finite relational structures, that is, sets equipped with relations, for example, hypergraphs. These two distinct developments were brought together in a common generalization due to Prömel [9], who proved a Ramsey theorem for finite parameter sets appropriately enriched to form relational structures. (We refer the reader to point 2 of Section 4 for the statement of Prömel's theorem in the terminology adopted in this paper.)

The two main goals of the present paper are as follows. First, frequently in mathematics structures are equipped not only with relations but also with functions, for example, algebras in the sense of universal algebra or structures in the sense of model theory. We give a generalization of Prömel's theorem to situations where structures have both relations and functions, with relations interpreted in a way specific to Ramsey theory. (We show how to derive Prömel's theorem from our Theorem 1 in point 2 of Section 4.) Second, if we disregard the above generalization, the argument given in the present paper provides a new proof of Prömel's theorem. For more on the proof of our theorem see the introductory paragraph of Section 3.

Date: July 30, 2008.

2000 *Mathematics Subject Classification.* 05C55, 05D10.

Key words and phrases. Ramsey theory, Prömel's theorem, finite structures.

Research supported by NSF grant DMS-0700841.

Filename: ramseystruct0708.

Recently deep connections have been found [2], [4], [8] between finite Ramsey theory (in particular, the Ramsey theorems mentioned above) and topological dynamics (in particular, extreme amenability). These connections with topological dynamics prompted the research of this paper and I hope that the methods presented here will make further applications in this area possible.

In Section 2, we formally introduce all the notions needed to state and prove our theorem in full generality, and we state the theorem. We follow this section with Section 3 containing a proof of the theorem. We have to make some choices in our presentation (for example, we use rigid surjections rather than parameter sets). We make them so that the notions we use fit naturally our arguments and give what seems to be the most general result, in the present context, that can be proved with our methods. In Section 4, we indicate how to reformulate statements about rigid surjections to obtain statements about parameter sets, and we comment on relations of our main result to the theorems of Prömel and of Abramson and Harrington and Nešetřil and Rödl.

Conventions. A natural number n is identified with the set of its predecessors $\{0, \ldots, n-1\}$ with the natural linear ordering on them. In particular, $0 = \emptyset$. For a set A, $|A|$ stands for its cardinality. By a d-coloring we understand a coloring with d many colors.

2. Definitions and statement of the theorem

Linear orders and rigid surjections. If K and L are linear orders, we call a function $s : L \to K$ *a rigid surjection* if it is surjective and the images of initial segments of L are initial segments of K. The composition of two rigid surjections is a rigid surjection.

The definition of rigid surjections comes from [10, p.164] and corresponds to the notion of partition; see also point 1 in Section 4. We point out that the notion of rigid surjection can be obtained by "dualizing" the notion of an increasing injection. If K and L are linear orders, $f : K \to L$ is an increasing injection if it is injective and the preimages of initial segments in L are initial segments in K. Rigid surjections are obtained by reversing the arrow in $K \to L$ and changing injection to surjection and preimages to images.

Bi-structures and epimorphisms. Bi-structures defined below are much like model theoretic structures in that function symbols and relation symbols are interpreted in them. However, while function symbols are interpreted as in model theory, relation symbols are interpreted differently. This is an important feature of the theory already present, in a different setup and in a slightly smaller generality, in Prömel's paper [9].

Let $\mathcal{L}$ be a set of symbols, called a language, consisting of relation symbols and function symbols. The language $\mathcal{L}$ can be empty. Each symbol in $\mathcal{L}$ has a non-negative integer associated with it, which is called the arity of the symbol. By an *$\mathcal{L}$-bi-structure* we understand a non-empty set X along with interpretations of symbols from $\mathcal{L}$, which is implemented as follows:

— with each function symbol $F \in \mathcal{L}$ of arity r we associated a function $F^X : X^r \to X$, where X^r represents the set of all functions from r to X;

— with each relation symbol $R \in \mathcal{L}$ of arity r we associated a set $R^X \subseteq r^X$, where r^X represents the set of all functions from X to r.

Recall that here a natural number r is identified with the set of its predecessors, that is, $r = \{0, \ldots, r-1\}$.

For two $\mathcal{L}$-bi-structures X, Y, a function $f : X \to Y$ is called an *epimorphism* if it is surjective, for each function symbol $F \in \mathcal{L}$ of arity r and $\eta \in X^r$, we have

$$F^Y(f \circ \eta) = f(F^X(\eta)), \tag{2.1}$$

and for each relation symbol $R \in \mathcal{L}$ of arity r and each $\gamma \in r^Y$, we have

$$\gamma \in R^Y \iff \gamma \circ f \in R^X. \tag{2.2}$$

We will write $R^X(\gamma)$ for $\gamma \in R^X$.

Ramsey theorem for linearly ordered bi-structures. Our aim is to prove the following theorem.

Theorem 1. *Let $\mathcal{L}$ be a language. Let $d > 0$ be a natural number. Let K, L be linear orders that are also $\mathcal{L}$-bi-structures. There exists a linear order M that is an $\mathcal{L}$-bi-structure such that*

(i) *for each d-coloring of the set of all epimorphic rigid surjections from M to K there exists an epimorphic rigid surjection $f_0 : M \to L$ such that*

$$\{f \circ f_0 : f : L \to K \text{ an epimorphic rigid surjection}\}$$

is monochromatic;

(ii) *each pair of distinct points of M is mapped by some epimorphic rigid surjection to two distinct points of L.*

Point (ii) of the theorem above allows us to transfer some properties of L to M. For instance, if a certain identity involving interpretations of function symbols holds in L, then point (ii) implies that it holds in M as well.

3. Proof of the bi-structural Ramsey theorem

A very flexible and influential method for proving Ramsey-type theorems for structures was developed by Nešetřil and Rödl. Constructions using this method consist of two separate amalgamation procedures: the first one called the partite lemma and the second one called the partite construction. (The adjective "partite" comes from the type of objects, called partite structures, that are used in these procedures.) Various applications of the above method differ essentially only in the first part. In the present paper, we follow this general approach in spirit, however, we will not use partite structures and our procedures will not be amalgamations. The main new contribution is again the first part of the proof (Lemma 3), in which rather than building the desired object by amalgamating small objects, we start with a large object and obtain the desired object as its subset. Additionally, we do not use, as is usual, partite structures, instead we replace them with certain combinatorial objects, which are generalizations of parameter sets.

In order to prove Theorem 1, it will be necessary to show a more general, but also more technical, result—Theorem 2. First we will introduce the notions needed to formulate this more general result. We will then devote the remainder of this section to its proof. We will be careful about details in this argument.

Objects and functions between them. Let $\mathcal{L}$ be a language. *Objects* are pairs (X, K) where X is a finite $\mathcal{L}$-bi-structure and K is a linear order with $K \subseteq X$. An object is called *linear* if $X = K$.

Let $\mathcal{X} = (X, K)$ and $\mathcal{Y} = (Y, L)$ be objects. A function $f : Y \to X$ is called a *epimorphism from $\mathcal{Y}$ to $\mathcal{X}$* if f is an epimorphism from Y to X, $f(L) = K$, and $f \restriction L$ is a rigid surjection from L to K. So being an epimorphism from one object to another requires more than just being an epimorphism between the underlying bi-structures. Let

$$\binom{\mathcal{Y}}{\mathcal{X}}_E$$

be the set of all epimorphisms from $\mathcal{Y}$ to $\mathcal{X}$. If f_1 and f_2 are epimorphisms from $\mathcal{Y}$ to $\mathcal{X}$ and from $\mathcal{Z}$ to $\mathcal{Y}$, respectively, then $f_1 \circ f_2$ is an epimorphism from $\mathcal{Z}$ to $\mathcal{X}$. If $f_0 \in \binom{\mathcal{Z}}{\mathcal{Y}}_E$, let

$$\binom{\mathcal{Y}}{\mathcal{X}}_E \circ f_0 = \{f \circ f_0 : f \in \binom{\mathcal{Y}}{\mathcal{X}}_E\} \subseteq \binom{\mathcal{Z}}{\mathcal{X}}_E.$$

We point out that the notion of object can be obtained by "dualizing" the notion of partite structures. Recall that a partite structure can be viewed as a structure together with a surjection from it onto a linear order. An object can be viewed as a bi-structure together with an injection from a linear order into it.

Ramsey theorem for objects. We fix a language $\mathcal{L}$ for the remainder of this section.

Theorem 2. *Let $d > 0$ be a given natural number. Let $\mathcal{X}$ be a linear object, and let $\mathcal{Y}$ be an object. Then there exists an object $\mathcal{Z}$ such that*

(i) *for any d-coloring of $\binom{\mathcal{Z}}{\mathcal{X}}_E$ there exists $f_0 \in \binom{\mathcal{Z}}{\mathcal{Y}}_E$ such that*

$$\binom{\mathcal{Y}}{\mathcal{X}}_E \circ f_0$$

is monochromatic;

(ii) *for distinct points z_1, z_2 in Z there is an epimorphism from $\mathcal{Z}$ to $\mathcal{Y}$ mapping z_1 and z_2 to distinct points.*

To derive Theorem 1 from the theorem above, fix two $\mathcal{L}$-bi-structures K and L that are also linear orders and a natural number $d > 0$. Let $\mathcal{X} = (K, K)$ and $\mathcal{Y} = (L, L)$ be the linear objects obtained by viewing the first element of each pair as and $\mathcal{L}$-bi-structure and the second one as a linear order. After applying Theorem 2 to these objects, we obtain an object $\mathcal{Z} = (Z, M')$. Now, we linearly order Z into a linear order M so that M' forms an initial segment of M and make M into an $\mathcal{L}$-bi-structure by retaining the interpretations of symbols from Z. It is easy to check that so defined M fulfills the conclusion of Theorem 1.

Proof of Theorem 2. We fix first some notation concerning Hales–Jewett lines that will be useful in what follows. Given a natural number N and a finite set P, by a *line ℓ in P^N* we understand a pair $\ell = (\bar{e}, u)$, where $u \subseteq N$ is non-empty and $\bar{e} = (e_k)_{k \in N \setminus u} \in P^{N \setminus u}$. We denote the set u by

$$d(\ell),$$

and for $k \in N \setminus d(\ell)$ we let

$$(\ell)_k = e_k.$$

For $\bar{f} \in P^N$, we write, with some abuse of notation,

$$\bar{f} \in \ell$$

if $\bar{f} \upharpoonright (N \setminus u) = \bar{e}$ and $\bar{f} \upharpoonright u$ is constantly equal to an element of P. This fixed value of the sequence $\bar{f} \upharpoonright u$ will be denoted by

$$\ell(\bar{f}).$$

Let $\mathcal{X} = (X, K)$, $\mathcal{Y} = (Y, L)$ be objects, and let $s_0 : L \to K$ be a rigid surjection from L to K. We write

$$\binom{\mathcal{Y}, s_0}{\mathcal{X}}_E$$

for the set of all epimorphisms f from $\mathcal{Y}$ to $\mathcal{X}$ such that $f \restriction L = s_0$.

Lemma 3. *Let $d > 0$ be given. Let $\mathcal{X}$ be a linear object whose linear order is K, let $\mathcal{Y} = (Y, M)$ be an object, and let $s_0 : M \to K$ be a rigid surjection. Then there exists an object $\mathcal{Z} = (Z, M)$ such that*

(i) *for any d-coloring of $\binom{\mathcal{Z}, s_0}{\mathcal{X}}_E$ there exists $f_0 \in \binom{\mathcal{Z}}{\mathcal{Y}}_E$ such that*

$$\binom{\mathcal{Y}, s_0}{\mathcal{X}}_E \circ f_0$$

is monochromatic;

(ii) *for distinct points z_1, z_2 in Z there is an epimorphism from $\mathcal{Z}$ to $\mathcal{Y}$ mapping z_1 and z_2 to distinct points.*

Proof. Let

$$P = \binom{\mathcal{Y}, s_0}{\mathcal{X}}_E .$$

(Note that the argument below works even in the case $P = \emptyset$.) Fix a large natural number N so that the Hales–Jewett theorem [5, p.1338] holds for P^N for d colors, and let

$$Q = \{\ell : \ell = (\bar{e}, u),\, \emptyset \neq u \subseteq N,\, \bar{e} \in P^{N \setminus u}\},$$

that is, Q is the set of all lines in P^N. Consider the following subset Z of Y^Q

$$(y_\ell)_{\ell \in Q} \in Z \iff$$
$$\forall \ell_1, \ell_2 \in Q \forall \bar{f} \; \big((\bar{f} \in \ell_1 \text{ and } \bar{f} \in \ell_2) \Rightarrow \ell_1(\bar{f})(y_{\ell_1}) = \ell_2(\bar{f})(y_{\ell_2}) \big).$$

The set Z will be the underlying set of the object (Z, M) as in the conclusion of the lemma. We will need to specify how the linear order M is placed inside of Z and how the symbols from $\mathcal{L}$ are interpreted in Z. Before doing this, we will define a number of auxiliary functions.

First define $\mu : M \to Y^Q$ by letting for $\ell \in Q$

$$\mu(x)_\ell = x. \tag{3.1}$$

So $\mu(x) = (\mu(x)_\ell)_{\ell \in Q}$ is the sequence indexed by $\ell \in Q$ constantly equal to x. Note that for any $\ell_0 \in Q$ and $\bar{f} \in \ell_0$, we have $\ell_0(\bar{f})(\mu(x)_\ell) = s_0(x)$. It follows that $\mu(x) \in Z$ so, in fact, $\mu : M \to Z$. Clearly μ is injective.

Let $k < N$. We define $\nu_k : Y \to Y^Q$ by describing, for a given $y \in Y$, a sequence $(\nu_k(y)_\ell)_{\ell \in Q}$. Fix $\ell \in Q$, and let

$$\nu_k(y)_\ell = \begin{cases} y, & \text{if } k \in d(\ell); \\ \min\{x \in M : s_0(x) = (\ell)_k(y)\}, & \text{if } k \notin d(\ell). \end{cases}$$

We claim that the range of ν_k is included in Z so, in fact, $\nu_k : Y \to Z$. To show that $\nu_k(y)$ defined above is in Z fix $\ell_1, \ell_2 \in Q$ and $\bar{f}$ such that $\bar{f} \in \ell_1$ and $\bar{f} \in \ell_2$. Set $y_{\ell_1} = \nu_k(b)_{\ell_1}$ and $y_{\ell_2} = \nu_k(y)_{\ell_2}$, and let f_k stand for the k-th entry in the sequence $\bar{f}$. We need to see that $\ell_1(\bar{f})(y_{\ell_1}) = \ell_2(\bar{f})(y_{\ell_2})$. By the obvious symmetry, three cases need to be considered:

$$k \in d(\ell_1) \cap d(\ell_2),\ k \in d(\ell_1) \setminus d(\ell_2),\ \text{and } k \notin d(\ell_1) \cup d(\ell_2).$$

In the first case,

$$\ell_1(\bar{f})(y_{\ell_1}) = f_k(y) = \ell_2(\bar{f})(y_{\ell_2}).$$

For the other two cases set

$$x_0 = \min\{x \in M : s_0(x) = (\ell_2)_k(y)\}.$$

Observe that for any $f \in \binom{\mathcal{Y}, s_0}{\mathcal{X}}$ we have $f(x_0) = s_0(x_0)$. With this in mind and with the above notation for x_0, in the second case, we have

$$\ell_1(\bar{f})(y_{\ell_1}) = f_k(y) = (\ell_2)_k(y) = s_0(x_0) = \ell_2(\bar{f})(x_0) = \ell_2(\bar{f})(y_{\ell_2}).$$

Finally, in the third case, we get

$$\ell_1(\bar{f})(y_{\ell_1}) = \ell_1(\bar{f})(x_0) = s_0(x_0) = \ell_2(\bar{f})(x_0) = \ell_2(\bar{f})(y_{\ell_2}).$$

Thus, $\nu_k(y)$ is in Z.

Finally, for $\ell_0 \in Q$, define a function $f_{\ell_0} : Z \to Y$ by letting

$$f_{\ell_0}((y_\ell)_{\ell \in Q}) = y_{\ell_0}. \tag{3.2}$$

We note for future reference that for $y \in Y$

$$f_{\ell_0}(\nu_k(y)) = y, \quad \text{if } k \in d(\ell_0). \tag{3.3}$$

Now we describe how the linear order M is placed inside of Z: we use the function μ defined above and its injectivity to identified M with the range of μ.

Our object (Z, M) will be completely defined as soon as we interpret in Z the symbols from $\mathcal{L}$, which we do as follows. Note that for a function symbol $F \in \mathcal{L}$ of arity r and $(y^1_\ell)_{\ell \in Q}, \ldots, (y^r_\ell)_{\ell \in Q}$ each taken from Z, we have that $(F^Y(y^1_\ell, \ldots, y^r_\ell))_{\ell \in Q}$ is an element of Z. This is so since, for any

$\ell_1, \ell_2 \in Q$ and $\bar{f}$ with $\bar{f} \in \ell_1$ and $\bar{f} \in \ell_2$, $\ell_1(\bar{f})$ and $\ell_2(\bar{f})$ are epimorphisms from Y to K, and therefore

$$\begin{aligned}\ell_1(\bar{f})(F^Y(y^1_{\ell_1}, \dots, y^r_{\ell_1})) &= F^K(\ell_1(\bar{f})(y^1_{\ell_1}), \dots, \ell_1(\bar{f})(y^r_{\ell_1})) \\ &= F^K(\ell_2(\bar{f})(y^1_{\ell_2}), \dots, \ell_2(\bar{f})(y^r_{\ell_2})) \\ &= \ell_2(\bar{f})(F^Y(y^1_{\ell_2}, \dots, y^r_{\ell_2})).\end{aligned}$$

Thus, we can define for $((y^1_\ell)_{\ell\in Q}, \dots, (y^r_\ell)_{\ell\in Q}) \in Z$

$$F^Z((y^1_\ell)_{\ell\in Q}, \dots, (y^r_\ell)_{\ell\in Q}) = (F^Y(y^1_\ell, \dots, y^r_\ell))_{\ell\in Q}.$$

We will now describe the interpretation of the relation symbols from $\mathcal{L}$. Let R be such a symbol of arity r. We have the following claim.

Claim. Let $\ell_1, \ell_2 \in Q$ and let $\gamma_1, \gamma_2 : Y \to r$. The equality $\gamma_1 \circ f_{\ell_1} = \gamma_2 \circ f_{\ell_2}$ implies that $R^Y(\gamma_1)$ if and only if $R^Y(\gamma_2)$.

Proof of Claim. Assume that $\gamma_1 \circ f_{\ell_1} = \gamma_2 \circ f_{\ell_2}$. To prove the claim we need to consider two cases. First assume that there exists $k \in d(\ell_1) \cap d(\ell_2)$. Fix such a k. Then, by (3.3), for each $y \in Y$ we have

$$\gamma_1(y) = \gamma_1 \circ f_{\ell_1}(\nu_k(y)) = \gamma_2 \circ f_{\ell_2}(\nu_k(y)) = \gamma_2(y).$$

Thus, $\gamma_1 = \gamma_2$.

Now assume $d(\ell_1) \cap d(\ell_2) = \emptyset$. Pick $k_1 \in d(\ell_1)$ and $k_2 \in d(\ell_2)$. Note that by our assumption $k_2 \notin d(\ell_1)$ and $k_1 \notin d(\ell_2)$, so we can set $g_1 = (\ell_1)_{k_2}$ and $g_2 = (\ell_2)_{k_1}$. We claim that for $y_1, y_2 \in Y$,

$$g_1(y_1) = g_2(y_2) \Longrightarrow \gamma_1(y_1) = \gamma_2(y_2). \tag{3.4}$$

Assume then that $g_1(y_1) = g_2(y_2)$, and let

$$x_0 = \min\{x \in M : g_1(y_1) = g_2(y_2) = s_0(x)\}.$$

Note that by definition of ν_{k_1} and ν_{k_2} we have

$$f_{\ell_2}(\nu_{k_1}(y_2)) = x_0 = f_{\ell_1}(\nu_{k_2}(y_1)).$$

From this formula and from (3.3), we get

$$\begin{aligned}\gamma_1(y_2) &= \gamma_1 \circ f_{\ell_1}(\nu_{k_1}(y_2)) = \gamma_2 \circ f_{\ell_2}(\nu_{k_1}(y_2)) = \gamma_2(x_0) \\ \gamma_2(y_1) &= \gamma_2 \circ f_{\ell_2}(\nu_{k_2}(y_1)) = \gamma_1 \circ f_{\ell_1}(\nu_{k_2}(y_2)) = \gamma_1(x_0).\end{aligned} \tag{3.5}$$

By definition of μ we get

$$\gamma_1(x_0) = \gamma_1 \circ f_{\ell_1}(\mu(x_0)) = \gamma_2 \circ f_{\ell_2}(\mu(x_0)) = \gamma_2(x_0). \tag{3.6}$$

By combining (3.5) with (3.6) we have $\gamma_1(y_1) = \gamma_2(y_2)$ as required.

It follows from (3.4) and from surjectivity of g_1 and g_2 that there exists $\gamma : K \to r$ such that $\gamma_1 = \gamma \circ g_1$ and $\gamma_2 = \gamma \circ g_2$. Since g_1 and g_2 are epimorphisms, we have

$$R^Y(\gamma_1) \Leftrightarrow R^K(\gamma) \Leftrightarrow R^Y(\gamma_2),$$

which is the conclusion of the claim.

For $\gamma' : Z \to r$, define $R^Z(\gamma')$ to hold if and only if there exists $\ell \in Q$ and $\gamma : Y \to r$ such that $R^Y(\gamma)$ and $\gamma' = \gamma \circ f_\ell$. The claim implies that this definition does not depend on the choice of ℓ and γ for a given γ' (which is important below in checking that each f_ℓ is an epimorphism).

The definitions above describe the $\mathcal{L}$-bi-structure Z, and hence the object (Z, M).

Let $\ell \in Q$. We claim that f_ℓ is an epimorphism from $\mathcal{Z}$ to $\mathcal{Y}$. Clearly from (3.1) we get that $f_\ell \circ \mu(x) = x$ for $x \in M$. Further, from the definition of the interpretation of the symbols from $\mathcal{L}$ in Z, we see that we only need to check that f_ℓ is surjective. This property follows from (3.3) applied to a $k \in d(\ell)$.

Note now that point (ii) of the lemma is witnessed by the epimorphisms f_ℓ with $\ell \in Q$.

It remains to check point (i). For each $\bar{e} \in P^N$ we define an epimorphism $g_{\bar{e}}$ from $\mathcal{Z}$ to $\mathcal{X}$ as follows. Let $\ell_0 \in Q$ be such that $\bar{e} \in \ell_0$. For $(y_\ell)_{\ell \in Q} \in Z$, let

$$g_{\bar{e}}((y_\ell)_{\ell \in Q}) = \ell_0(\bar{e})(y_{\ell_0}).$$

Note that the definition of Z insures that $g_{\bar{e}}$ does not depend on the choice of ℓ_0. Note further that

$$g_{\bar{e}} = \ell_0(\bar{e}) \circ f_{\ell_0}, \tag{3.7}$$

hence $g_{\bar{e}}$, being the composition of two epimorphisms, it is an epimorphism.

Fix a d-coloring c of $\binom{\mathcal{Z}, s}{\mathcal{X}}$. Then c induces a d-coloring of P^N by

$$P^N \ni \bar{e} \to c(g_{\bar{e}}).$$

By the Hales–Jewett theorem, we can fix $\ell_0 \in Q$ such that the above coloring is constant on all $\bar{e} \in P^N$ with $\bar{e} \in \ell_0$. We now claim that the set

$$\binom{\mathcal{Y}, s_0}{\mathcal{X}}_E \circ f_{\ell_0}$$

is monochromatic. Indeed, let $h \in \binom{\mathcal{Y}, s_0}{\mathcal{X}}$, and define $\bar{e}_h \in P^N$ by requiring that $(\bar{e}_h)_k = h$ if $k \in d(\ell_0)$, and $(\bar{e}_h)_k = (\ell_0)_k$ if $k \notin d(\ell_0)$. Then, by (3.7),

$$h \circ f_{\ell_0} = \ell_0(\bar{e}_h) \circ f_{\ell_0} = g_{\bar{e}_h}.$$

Thus, $c(h \circ f_{\ell_0}) = c(g_{\bar{e}_h})$, which does not depend on h since $\bar{e}_h \in \ell_0$. It follows that (i) holds. □

Proof of Theorem 2. Let $\mathcal{X} = (K, K)$ and $\mathcal{Y} = (Y, L)$. Let M be a linear order obtained by applying the Graham–Rothschild theorem [5, p.1361] to linear orders K and L with d colors. So for any d-coloring of all rigid surjections from M to K there exists a rigid surjection $j : M \to L$ such that the same color is assigned to all $s \circ j$ as s runs over the set of all rigid surjection from L to K. (This formulation of the Graham–Rothschild theorem, which uses rigid surjections rather than partitions, will be applied later on in the proof. See Section 4 point 1.)

Define

$$Y_0 = Y^{\binom{M}{L}_E}.$$

Set $h_j : Y_0 \to Y$ to be the projection on the j-th coordinate for $j \in \binom{M}{L}_E$. We identify the linear order M with a subset of Y_0 via the following injection $\mu_0 : M \to Y_0$ given by $\mu_0(x) = (\mu(x)_j)_j$ with

$$\mu_0(x)_j = j(x)$$

where j runs over $\binom{M}{L}_E$. For a function symbol $F \in \mathcal{F}$ of arity r we let

$$F^{Y_0}((y_j)_j) = (F^Y(y_j))_j.$$

For a relation symbol $R \in \mathcal{L}$ of arity r, for $\gamma \in r^{Y_0}$ we let $R^{Y_0}(\gamma)$ if and only if there exists $\gamma' \in r^Y$ and $j \in \binom{M}{L}_E$ such that $R^Y(\gamma')$ and $\gamma = \gamma' \circ f_j$. Note that if $\gamma_1' \circ f_{j_1} = \gamma_2' \circ f_{j_2}$, then $\gamma_1' = \gamma_2'$. (If $j_1 \neq j_2$, then additionally both γ_1' and γ_2' are constant functions.) Thus, the definition of R^{Y_0} does not depend on the choice of γ' and j.

Note that each f_j is an epimorphism from $\mathcal{Y}_0$ to $\mathcal{Y}$.

Let $\{s_q : 1 \leq q \leq t\}$ be an enumeration of all rigid surjections from M to K. We now define objects $\mathcal{Y}_q$ for $0 \leq q \leq t$. The object $\mathcal{Y}_0$ has already been constructed. If $\mathcal{Y}_q$ is defined, let $\mathcal{Y}_{q+1}$ be the object obtained in Lemma 3 when applied to $\mathcal{X}$, $\mathcal{Y}_q$, and s_q. Define

$$\mathcal{Z} = \mathcal{Y}_t.$$

Note that point (ii) of the theorem holds for $\mathcal{Y}_0$. One easily checks by induction that for any $0 \leq q < t$ if this property holds for $\mathcal{Y}_q$, it holds for $\mathcal{Y}_{q+1}$, which verifies point (ii) of the theorem.

To see point (i), let c be a k-coloring of $\binom{\mathcal{Y}_t}{\mathcal{X}}_E$. By considering the restriction of c to $\binom{\mathcal{Y}_t, s_t}{\mathcal{X}}_E$, we obtain $f_t \in \binom{\mathcal{Y}_t}{\mathcal{Y}_{t-1}}_E$ from Lemma 3(i). Having produced

$$f_t \in \binom{\mathcal{Y}_t}{\mathcal{Y}_{t-1}}_E, \dots, f_{q+1} \in \binom{\mathcal{Y}_{q+1}}{\mathcal{Y}_q}_E,$$

with $q \geq 1$, consider the restriction of c to $\binom{\mathcal{Y}_q, s_q}{\mathcal{X}}_E \circ f_{q+1} \circ \dots \circ f_t$, and apply Lemma 3 to this d-coloring obtaining $f_q \in \binom{\mathcal{Y}_q}{\mathcal{Y}_{q-1}}_E$ such that $\binom{\mathcal{Y}_{q-1}, s_q}{\mathcal{X}}_E \circ$

$f_q \circ f_{q+1} \circ \cdots \circ f_t$ is monochromatic with respect to c. This way we obtain $f_t, \ldots, f_1$.

Now, it is easy to check that for $g \in \binom{\mathcal{Y}_0}{\mathcal{X}}_E$, the value of

$$c(g \circ f_1 \circ \cdots \circ f_q)$$

depends only on the rigid surjection $g \restriction M : M \to K$. To see this let $g_1, g_2 \in \binom{\mathcal{Y}_0}{\mathcal{X}}_E$ be such that $g_1 \restriction M = g_2 \restriction M = s_q$ for some $1 \le q \le t$. Then, since

$$g_1 \circ f_1 \circ \cdots \circ f_{q-1},\, g_2 \circ f_1 \circ \cdots \circ f_{q-1} \in \binom{\mathcal{Y}_{q-1}, s_q}{\mathcal{X}}_E,$$

we get

$$\begin{aligned} c(g_1 \circ f_1 \circ \cdots \circ f_t) &= c((g_1 \circ f_1 \circ \cdots \circ f_{q-1}) \circ f_q \circ \cdots \circ f_t) \\ &= c((g_2 \circ f_1 \circ \cdots \circ f_{q-1}) \circ f_q \circ \cdots \circ f_t) = c(g_2 \circ f_1 \circ \cdots \circ f_t). \end{aligned}$$

Thus, we can consider a d-coloring c' of rigid surjections from M to K which is an arbitrary extension of the function

$$g \restriction M \to c(g \circ f_1 \circ \cdots \circ f_t),$$

for $g \in \binom{\mathcal{Y}_0}{\mathcal{X}}_E$. By the choice of M and the construction of $\mathcal{Y}_0$, there exists $j \in \binom{M}{L}_E$ such that the set

$$\binom{\mathcal{Y}}{\mathcal{X}}_E \circ (h_j \restriction M)$$

is monochromatic with respect to c'. Thus, the set

$$\binom{\mathcal{Y}}{\mathcal{X}}_E \circ h_j \circ f_1 \circ \cdots \circ f_t$$

is monochromatic with respect to c. It follows that

$$f_0 = h_j \circ f_1 \circ \cdots \circ f_t \in \binom{\mathcal{Z}}{\mathcal{Y}}_E$$

witnesses that point (i) of the theorem holds for the coloring c. □

4. Comments and reformulations

In point 1 of this section, we indicate how to translate statements involving partitions and parameter sets into statements about rigid surjections. Further, in point 2, we state Prömel's theorem and show how to derive it from Theorem 1. Note that Prömel's theorem for partitions follows from Theorem 1 for languages with relation symbols only, while the derivation of Prömel's theorem for the more general setting of parameter sets involves languages with function symbols (of arity 0, that is, constant symbols). In point 3, we comment on the relation of Prömel's theorem to the results of [1], [6], and [7].

1. Graham and Rothschild's and Prömel's theorems are usually stated in terms of partitions or, more generally, A-parameter sets, where A is a finite set. (Partitions are $\emptyset$-parameter sets.) As pointed out in [10, p.164], rigid surjections are in a functorial bijection with partitions. Given a rigid surjection $f : L \to K$ one obtains a partition $\mathcal{P}_f$ of L into $|K|$ preimages of points in K under f, and vice versa given a partition $\mathcal{P}$ of L into $|K|$ sets one obtains a rigid surjection $f_{\mathcal{P}} : L \to K$ by mapping each point of the element of $\mathcal{P}$ with the i-th minimal point to the i-th point of K. These two operations reverse each other. Moreover, composition of rigid surjections corresponds in a natural way to taking coarser partitions. Quite similarly, A-parameter sets are in a functorial bijection with those rigid surjections from a linear order L to a linear order K, both with at least $|A|$ elements, that map the i-th element of L to the i-th element of K for each $i \leq |A|$. We call such surjections $|A|$-rigid surjections.

2. We will explain now how Prömel's theorem follows from Theorem 1. First, we will restate Prömel's theorem using the notions from point 1 above. We ask the reader to consult [9, Theorem 4.1] for the original statement. In Prömel's theorem for A-parameter sets, one assumes the following convention: the language $\mathcal{L}$ contains only relation symbols of arity $r \geq |A|$, the underlying sets of $\mathcal{L}$-structures are linear orders K of size $\geq |A|$, and a relation symbol $R \in \mathcal{L}$ of arity r is interpreted as a subset of all $|A|$-rigid surjections from K to a linear order of size $r \geq |A|$. In this situation, we will identify this last linear order with $r = \{0, \dots, r-1\}$ with its natural order. (This notion of interpretation is more restrictive, but non-essentially so, than the one adapted by us in Section 2.) Once these interpretations are defined one defines epimorphisms as in formula (2.2).

Now Prömel's theorem can be translated into the following statement. *With the above conventions, given two linear orders K and L with $|K|, |L| \geq |A|$ on which symbols from $\mathcal{L}$ are interpreted as above, there exists a linear order M with interpretation of symbols in $\mathcal{L}$ as above so that for any d-coloring of all epimorphic $|A|$-rigid surjections from M to K there exists an epimorphic $|A|$-rigid surjection $f_0 : M \to L$ such that the set*

$$\{f \circ f_0 : f : L \to K \text{ is an epimorphic } |A|\text{-rigid surjection}\}$$

is monochromatic.

To obtain this statement from Theorem 1, consider the language $\mathcal{L}$ as above and augment it to a language $\mathcal{L}'$ by adding $|A|$ distinct constant symbols, that is, function symbols of arity 0. To make the description below easier, we linearly order the new constant symbols. Given two linear orders K and L that interpret symbols from $\mathcal{L}$ with the convention described above, we make them into $\mathcal{L}'$-bi-structures by interpreting the i-th constant symbol in $\mathcal{L}'$ as the i-th element in K and keeping the old interpretations of relation symbols from $\mathcal{L}$. We do the same with L. We now apply Theorem 1 and obtain an $\mathcal{L}'$-bi-structure M that is also a linear order. Note that Theorem 1(ii) implies that distinct constant symbols are interpreted in M as distinct elements. Let π be a permutation of M that preserves the order on M among all elements that are not interpretations of constants and maps the interpretation of the i-th constant symbol of $\mathcal{L}'$ to the i-th element of M. Now given an interpretation R^M of a relation symbol $R \in \mathcal{L}$ of arity r from the conclusion of Theorem 1 define a new interpretation of R by letting

$$R^{M,0} = \{\gamma : M \to r : \gamma \text{ is an } |A|\text{-rigid surjection and } \gamma \circ \pi \in R^M\}, \tag{4.1}$$

where in the expression $\gamma : M \to r$ we view r as a linear order of size r. With these new interpretations, the resulting structure fulfills the conclusion of Prömel's theorem. In fact, if f_0 is a function gotten from the conclusion of Theorem 1, then $f_0 \circ \pi^{-1}$ ensures that the conclusion of Prömel's theorem holds. We leave verifying details of this argument to the reader. We only point out as hints that if $f : M \to L$ is an $\mathcal{L}'$-epimorphism that is a rigid surjection, then $f \circ \pi^{-1} : M \to L$ is an $|A|$-rigid surjection, and that if $h : M \to L$ is an $|A|$-rigid surjection and $\gamma : L \to r$, then

$$\gamma \circ h \text{ is an } |A|\text{-rigid surjection } \iff \gamma \text{ is an } |A|\text{-rigid surjection.}$$

From the above two assertions it follows that if $f : M \to L$ is an $\mathcal{L}'$-epimorphism, then $f \circ \pi^{-1} : M \to L$ is an $\mathcal{L}$-epimorphism with the interpretation of relation symbols in M given by (4.1).

3. We recall that a derivation of the theorem of Abramson and Harrington [1] and Nešetřil and Rödl [6], [7] from Prömel's theorem is presented in Prömel's original paper [9, Corollary 4.3].

References

[1] F.G. Abramson, L.A. Harrington, *Models without indicernibles*, J. Symb. Logic 43 (1978), 572–600.

[2] E. Glasner, B. Weiss, *The universal minimal system for the group of homeomorphisms of the Cantor set*, Fund. Math. 176 (2003), 277–289.

[3] R.L. Graham, B.L. Rothschild, *Ramsey's theorem for n-parameter sets*, Trans. Amer. Math. Soc. 159 (1971), 257–292.

[4] A.S. Kechris, V.G. Pestov, S. Todorcevic, *Fraïssé limits, Ramsey theory, and topological dynamics of automorphism groups*, Geom. Funct. Anal. 15 (2005), 106–189.

[5] J. Nešetřil, *Ramsey theory*, in *Handbook of Combinatorics*, Elsevier, 1995.

[6] J. Nešetřil, V. Rödl, *Partition of relational and set systems*, J. Combin. Theory Ser. A 22 (1977), 289–312.

[7] J. Nešetřil, V. Rödl, *Ramsey classes of set systems*, J. Combin. Theory Ser. A 34 (1983), 183–201.

[8] V.G. Pestov, *On free actions, minimal flows, and a problem by Ellis*, Trans. Amer. Math. Soc. 350 (1998), 4149–4165.

[9] H.J. Prömel, *Induced partition properties of combinatorial cubes*, J. Combin. Theory Ser. A 39 (1985), 177–208.

[10] H.J. Prömel, B. Voigt, *Hereditary atributes of surjections and parameter sets*, Europ. J. Combinatorics 7 (1986), 161–170.

[11] F.P. Ramsey, *On a problem of formal logic*, Proc. London Math. Soc. 30 (1930), 264–286.

Department of Mathematics, University of Illinois, 1409 W. Green St., Urbana, IL 61801

E-mail address: ssolecki@math.uiuc.edu