THE COSET EQUIVALENCE RELATION AND TOPOLOGIES ON SUBGROUPS

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Abstract. The paper studies the structure of the homogeneous space G/H, for G a Polish group and $H < G$ a Borel, not necessarily closed subgroup of G, from the point of view of the theory of definable equivalence relations. It makes a connection between the complexity of the natural *coset equivalence relation* associated with G/H and *Polishability* of H, that is, the possibility of introducing a Polish group topology on H respecting its Borel structure. In particular, it is proved that if H is an Abelian Borel subgroup of a Polish group G, then either H is Polishable or $\mathbb{E}_1$ continuously embeds into the coset equivalence relation induced by H on G. The same conclusion is shown to hold if H is an increasing union of a sequence of Polishable subgroups of G.

1. Introduction

The following notion of reduction is fundamental in comparing the relative complexity of equivalence relations on Polish spaces. Given two equivalence relations E and F defined on Polish spaces X and Y, respectively, we say that E is *Borel reducible to* F, in symbols, $E \leq_B F$, if there exists a Borel function $f : X \to Y$ such that

$$x_1 E x_2 \Leftrightarrow f(x_1) F f(x_2). \tag{1.1}$$

This relationship between E and F can be rephrased in somewhat more intuitive terms by saying that there exits an injection from X/E to Y/F that is given by, or in other words can be lifted to, a Borel function from X to Y. Define the equivalence relation $\mathbb{E}_1$ on $(2^{\mathbb{N}})^{\mathbb{N}}$ by letting two sequences $(x_n), (y_n)$ of elements of $2^{\mathbb{N}}$ be $\mathbb{E}_1$-equivalent if $x_n = y_n$ for large enough n. Given an action of a group H on a set X, by the *orbit equivalence relation* we understand the equivalence relation on X induced by its partition into orbits of the action of H.

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Seen in a broader context, the results of this work are part of the program of investigating the structure of Borel equivalence relations. In this program, the class of all Borel equivalence relations is divided with the help of the relation $\leq_B$ through dichotomy theorems (partly proved, partly conjectural) into subclasses of mathematical importance. For more details of the picture of this vast lanscape we refer the reader to Section 2 of [12] and the literature cited in that paper. The results presented here explore the broadest conjectural dichotomy from [12]: given a Borel equivalence relation E on a Polish space, either E is Borel reducible to an orbit equivalence relation induced by a continuous action of a Polish group on a Polish space or else $\mathbb{E}_1 \leq_B E$.

On a technical level, the paper is about the natural translation action of a Borel subgroup H of a Polish group G on G:

$$G \times H \ni (g,h) \to gh^{-1} \in G. \tag{1.2}$$

The orbits of the action are left cosets of H in G and the orbit equivalence relation of this action is given by

$$g_1 E_{G/H} g_2 \Leftrightarrow g_2^{-1} g_1 \in H, \tag{1.3}$$

and is called the *coset equivalence relation*. We are interested in the complexity of the equivalence relation $E_{G/H}$ and its connections with the topology on H.

I will describe now more precisely the circumstances in which the problems considered in this paper came about. Before doing it, however, I will need to state important to us definitions. By a *Borel group* we understand a topological group whose topology is separable metric and which is a Borel subset of some (or, equivalently, each) of its metric completions. Obviously, Borel subgroups of Polish groups are Borel groups, but also the opposite inclusion holds: each Borel group is a subgroup of some Polish group. In fact, the ambient Polish group is uniquely determined if we assume that the Borel group is dense in it. A Borel group H is called *Polishable* if there exists a Polish group topology on it whose family of Borel sets is equal to the family of Borel subsets of H (with respect to the original topology on H). Such a Polish group topology is unique, if it exists, and contains the original topology on H. (For proofs of these folklore facts see the introduction of [21].) The notion of Polishability was introduced in [13]. It, or conditions analogous to it, show up in a variety of situations, see for example [6], [7], [10], [16], [19], [20], [21].

Kechris and Louveau [13, Theorem 4.2] proved that if a Polishable group H acts continuously on a Polish space X, then $\mathbb{E}_1$ does not Borel reduce to the orbit equivalence relation. This result naturally gives rise to problems concerning various versions of its converse; for some of them see [13, p.241]. One of them can be phrased as follows: is Borel reduction of $\mathbb{E}_1$ to the orbit equivalence relation of some continuous action of a Borel group H the only obstacle to H being Polishable? To answer this question in the affirmative, given a non-Polishable Borel group H one needs to construct a continuous action of H on a Polish space such that the induced orbit equivalence relation Borel reduces $\mathbb{E}_1$. Each Borel group H has a natural action which appears to be complicated if H is not Polishable and which is defined as follows. As mentioned above, there exists a unique Polish group G containing H as a dense subgroup and we can consider the translation action of H on G given by (1.2). The orbit equivalence relation is then given by (1.3). Therefore, the two theorems below imply that the converse to the Kechris–Louveau result holds if H is Abelian and if H is an increasing union of a sequence of Polishable groups. These are the main results of the present paper. To state their sharp versions, we will need a notion of inequality between equivalence relations stronger than $\leq_B$. Given two equivalence relations E and F on Polish spaces X and Y, respectively, we write $E \sqsubseteq_c F$ if there is a continuous and injective function $f : X \to Y$ fulfilling (1.1).

Theorem 1.1. *Let H be an Abelian Borel subgroup of a Polish group G. Then either H is Polishable or $\mathbb{E}_1 \sqsubseteq_c E_{G/H}$.*

Theorem 1.2. *Let G be a Polish group, and let H be a subgroup of G that is the union of an increasing sequence H_n, $n \in \mathbb{N}$, of Polishable subgroups of G. Then either for some n, H/H_n is countable, and so H is Polishable, or $\mathbb{E}_1 \sqsubseteq_c E_{G/H}$.*

Even though the two theorems above apply to different kinds of subgroups, they are both consequences of the same general results which give methods of proving Polishability of a subgroup and of embedding $\mathbb{E}_1$ into the coset equivalence relation. These general results are presented in Section 2 and proved in the subsequent sections. There are situations, see Example 2.8, which are not covered by Theorems 1.1 and 1.2 but are covered by the general results.

The following statements related to the theorems above were established earlier.

Let H be a Borel subgroup of a Polish group G. Either $\mathbb{E}_1 \sqsubseteq_c E_{G/H}$ or H is Polishable in the following three situations:

1. (Kechris–Louveau [13]) *G arbitrary Polish, $H = \bigcup_{n\in\mathbb{N}} H_n$ where $H_n \subseteq H_{n+1}$ with each H_n a closed subgroup of G;*
2. (Solecki [20]) *$G = 2^{\mathbb{N}}$ with the coordinatewise addition modulo 2 and H an ideal;*
3. (Casevitz [3]) *$G = \mathbb{R}^{\mathbb{N}}$ and H a linear subspace of $\mathbb{R}^{\mathbb{N}}$ with the additional property that if $(x_n) \in H$ and (y_n) is such that $|y_n| \leq x_n$ for all n, then $(y_n) \in H$.*

In point (1), Polishability of H is equivalent to the more concrete condition that H/H_n be countable for some n. Theorem 1.1 strengthens results (2) and (3) while Theorem 1.2 strengthens (1). One should mention, however, that the assumptions in (1)–(3) above are rather strong. For example, in (1) the assumptions make H $\mathbf{\Sigma}^0_2$; in (2) and (3), if H turns out to be Polishable, it is $\mathbf{\Pi}^0_3$, see [20] and [3]. These definability conditions fail for most groups covered by Theorems 1.1 and 1.2. Also, in situations (2) and (3), both conditions, $\mathbb{E}_1 \sqsubseteq_c E_{G/H}$ and H Polishable, imply further structural properties of H, see [20] and [3]. Again, this is not true for more general groups as in Theorems 1.1 and 1.2.

The related problem of explaining in terms of the complexity of $E_{G/H}$ when a Borel subgroup H of a Polish group G is Polish with the topology inherited from G was solved satisfactorily some time ago. To state this result, I will need some definitions. The equality equivalence relation on $2^{\mathbb{N}}$ is denoted by $\Delta_{2^{\mathbb{N}}}$. Define $\mathbb{E}_0$ to be the equivalence relation on $2^{\mathbb{N}}$ which makes two binary sequences $x, y \in 2^{\mathbb{N}}$ equivalent if they are eventually equal. The following result is a combination of a theorem due to Miller [17, Theorem 3], who generalized a result of Mackey [15], and a theorem of Becker and Kechris [2, Theorem 3.4.5], which generalizes earlier work of Effros [5].

Let H be a Borel subgroup of a Polish group G. The following conditions are equivalent:

(i) *H is Polish with the topology inherited from G;*
(ii) *$\mathbb{E}_0 \not\sqsubseteq_c E_{G/H}$;*
(iii) *$E_{G/H} \leq_B \Delta_{2^{\mathbb{N}}}$;*
(iv) *H is closed.*

Note the similarity between conditions (i) and (ii) above and the two conditions in the conclusion of Theorems 1.1 and 1.2. This similarity does not extend to the proofs.

2. Fundamental notions and general results

Before turning to the main topics of this section, we will recall some standard definitions and some folklore facts. Let H be a subgroup of a second countable group G. We say that H is *analytic* if it is a continuous image of $\mathbb{N}^{\mathbb{N}}$. All Borel subgroups of Polish groups are analytic. If τ is a Polish group topology on H, then the following conditions are equivalent

(a) τ has the same Borel sets as the ones H inherits from the inclusion $H \subseteq G$;
(b) each open subset of H in the topology inherited from G is Borel with respect to τ;
(c) τ contains the topology H inherits from G.

Further, in any of these circumstance H is a Borel subgroup of G. In the sequel, we will freely use the equivalence of the above conditions under the assumptions that H is Polishable.

We use the standard notation $\mathbf{\Sigma}^0_\alpha$ and $\mathbf{\Pi}^0_\alpha$, $1 \leq \alpha < \omega_1$, for additive and multiplicative levels of the Borel hierarchy. So $\mathbf{\Sigma}^0_1$ stands for open sets, $\mathbf{\Pi}^0_1$ for closed sets, $\mathbf{\Sigma}^0_2$ for F_σ sets, $\mathbf{\Pi}^0_3$ for $F_{\sigma\delta}$ sets, and so on.

For a definition of a tree on $\mathbb{N}$ and of a pruned such tree we refer the reader to [11]. If T is a tree on $\mathbb{N}$, $[T]$ stands for the set of all elements of $\mathbb{N}^{\mathbb{N}}$ that are branches through T.

2.1. **Canonical approximations.** Let G be a Polish group, let H be a subgroup of G, and let α be a countable ordinal. Let $(H_\xi, \tau_\xi)_{\xi<\alpha}$ be a decreasing transfinite sequence consisting of Polishable subgroups H_ξ of G with the Polish group topology τ_ξ on H_ξ containing the topology H_ξ inherits from G. We call such a sequence a *canonical approximation for H of length α* if the following conditions hold

(i) $H_0 = \overline{H}$;

and for each $\xi < \alpha$

(ii) $H < H_\xi$ and H is τ_ξ dense in H_ξ;
(iii) $H_\xi = \bigcap_m \bigcup_n F_{m,n}$, where for each m, n there exists $\gamma < \xi$ such that $F_{m,n}$ is τ_γ-closed;
(iv) if $A \subseteq G$ is such that $H \subseteq A$ and $A = \bigcap_m \bigcup_n F_{m,n}$, where for each m, n there exists $\gamma < \xi$ with $F_{m,n}$ τ_γ-closed, then $A \cap H_\xi$ is comeager with respect to τ_ξ.

If the length of a canonical approximation is a successor $\alpha + 1$, we sometimes write $(H_\xi, \tau_\xi)_{\xi\leq\alpha}$ for $(H_\xi, \tau_\xi)_{\xi<\alpha+1}$.

A version of the definition of canonical approximation for a *Polishable* H is implicit in [21, Theorem 2.1] and is made explicit in [6, p.501]. There is a small difference between that definition and the formulation given above. In [6], conditions (iii) and (iv) are asserted only for successor ξ, while for limit ξ, H_ξ is defined to be $\bigcap_{\gamma<\xi} H_\gamma$. It follows, however, from Lemma 2.3(ii) (with $H' = H$) and Lemma 2.3(iii) that the two versions of the definition are equivalent if H is Polishable.

We prove now some lemmas on canonical approximations. These lemmas will justify the statement in the preceding paragraph. More importantly, they will also be used in Subsection 2.4 below and in Section 4.

First we adopt a certain convention. We will frequently be given a subgroup S of a Polish group G with a group topology σ on S containing the topology S inherits from G. In this situation, given $A \subseteq G$ we let

$$\sigma\text{-int}(A) = \bigcup\{U : U \ \sigma\text{-open and } U \subseteq A\}.$$

Note that this definition makes sense, and will be used, for arbitrary subsets A of G and not only those contained in S.

The first lemma justifies the word "canonical" in the definition above.

Lemma 2.1. *Let G be Polish and let H be a subgroup of G. Let $(H_\xi, \tau_\xi)_{\xi<\alpha}$ and $(H'_\xi, \tau'_\xi)_{\xi<\beta}$ be two canonical approximations for H. Assume $\alpha \le \beta$. Then $H_\xi = H'_\xi$ and $\tau_\xi = \tau'_\xi$ for all $\xi < \alpha$.*

Proof. Since τ_ξ and τ'_ξ are uniquely determined by H_ξ and H'_ξ together with the topology they inherit from G, respectively, it suffices to show that $H_\xi = H'_\xi$ for $\xi < \alpha$. The proof proceeds by induction on ξ. There is no problem if $\xi = 0$. Assume now $H_\gamma = H'_\gamma$ for all $\gamma < \xi$. Since by (iii) both H_ξ and H'_ξ are of the form $\bigcap_m \bigcup_n F_{m,n}$ with each $F_{m,n}$ τ_γ-closed for some $\gamma < \xi$ and since, by (ii), they both contain H, we have by (iv) that $H_\xi \cap H'_\xi$ is comeager with respect to τ_ξ and with respect to τ'_ξ. Since by Pettis' theorem [11], a comeager subgroup of a Polish group is equal to the whole group, it follows that

$$H_\xi \cap H'_\xi = H_\xi = H'_\xi.$$

Thus, the conclusion is proved for ξ. □

The following technical lemma on stabilization of interiors will be useful in analyzing limit stages of canonical approximations in Lemma 2.3 and will be crucial in Section 4 in the proof of Lemma 4.1.

Lemma 2.2. *Let G be a Polish group, let H be a subgroup of G, and let $(H_\xi, \tau_\xi)_{\xi<\alpha}$ be a canonical approximation for $H < G$. Fix γ, ξ with $\gamma < \xi < \alpha$. Let $S < H_\xi$ have a second countable group topology σ containing the topology inherited by S from H_ξ taken with τ_ξ. Assume that $H \subseteq S$ and that H is σ-dense in S. Then for any F that is τ_γ-closed, we have*

$$\tau_\xi\text{-int}(F) \supseteq \sigma\text{-int}(F).$$

Proof. To prove the lemma, note that since all the relevant topologies are group topologies, it will suffice to show that for any τ_γ-closed set F with $1 \in \sigma\text{-int}(F)$, we have

$$1 \in \tau_\xi\text{-int}(F).$$

Since τ_ξ contains $\tau_{\gamma+1}$ restricted to H_ξ, it will be sufficient to prove

$$1 \in \tau_{\gamma+1}\text{-int}(F). \tag{2.1}$$

Let $1 \in U \subseteq F$ be σ-open. Find $W \subseteq S$ that is σ-open and such that $W^{-1}W \subseteq U$. Since H is σ-dense in S, there are $h_n \in H$, $n \in \mathbb{N}$, such that

$$\bigcup_n h_n W = S \supseteq H. \tag{2.2}$$

Let C be the closure of W with respect to τ_γ. It follows from (2.2) that $\bigcup_n h_n C$ is a $\mathbf{\Sigma}^0_2$ with respect to τ_γ subset of H_γ covering H. By point (iv) of the definition of canonical approximations, $\bigcup_n h_n C \cap H_{\gamma+1}$ is comeager with respect to $\tau_{\gamma+1}$. Since each set $h_n C \cap H_{\gamma+1}$ is $\tau_{\gamma+1}$-closed, one of them, say $h_{n_0} C \cap H_{\gamma+1}$, has non-empty interior with respect to $\tau_{\gamma+1}$. Thus,

$$1 \in \tau_{\gamma+1}\text{-int}((h_{n_0}C)^{-1}h_{n_0}C) \subseteq C^{-1}C \subseteq F,$$

and (2.1) follows. □

The following lemma analyzes the limit stages of a canonical approximation. Points (ii) and (iii) in it were used above to justify the equivalence of the definition of canonical approximation presented here with a version of it from [21] and [6]. Further, it will be used in Subsection 2.4 below.

Lemma 2.3. *Let G be a Polish group, and let H be a subgroup of G with a canonical approximation $(H_\xi, \tau_\xi)_{\xi<\alpha}$.*

(i) *If α is limit, then $\bigcap_{\xi<\alpha} H_\xi$ is a Polishable group with a unique Polish group topology containing the topology this group inherits from G and in which H is dense.*

(ii) *Let α be limit. Put $H_\alpha = \bigcap_{\xi<\alpha} H_\xi$, and let τ_α be the Polish group topology on H_α guaranteed to exist by (i). Assume there exists a Polishable subgroup H' of H_α with a Polish group topology τ' such that $H < H'$ and $B \cap H'$ is τ'-comeager for any set B which contains H and which is the countable union of sets each of which is closed with respect to some τ_ξ with $\xi < \alpha$. Then $(H_\xi, \tau_\xi)_{\xi\leq\alpha}$ is a canonical approximation for H.*

(iii) *If $\lambda < \alpha$ is limit, then $H_\lambda = \bigcap_{\xi<\lambda} H_\xi$.*

Proof. (i) Let $\xi_n < \xi_{n+1} < \alpha$, $n \in \mathbb{N}$, be a sequence cofinal in α. Let d_n be a complete metric on H_{ξ_n} compatible with τ_{ξ_n} and bounded by 1. Note that for each n, $H_{\xi_{n+1}} \subseteq H_{\xi_n}$ and the topology induced on $H_{\xi_{n+1}}$ by the restriction of d_n to it is contained in the topology induced on it by d_{n+1}.

For $h_1, h_2 \in H$, let

$$d(h_1, h_2) = \sum_{n\in\mathbb{N}} 2^{-n} d_n(h_1, h_2).$$

Clearly d is a metric on H. It induces a group topology, since the restrictions of the metrics d_n to H induce group topologies on H. The topology induced by d contains the topology H inherits from G, since this is true about the restriction of d_0 to H. To finish the proof of the lemma, it suffices to show that d is complete. By the definition of d, a sequence (h_i) of elements of H is d-Cauchy if and only if it is d_n-Cauchy for each n, which is equivalent, by completeness of each d_n, to d_n-convergence of (h_i) in H_{ξ_n} for each n. By the remark from the first paragraph of this proof, for each n, the d_{n+1}-limit of (h_i) in $H_{\xi_{n+1}}$ is equal to the d_n limit of (h_i). It follows that all these limits are equal to some $h \in G$. Since h is in each H_{ξ_n} and since the sequence (ξ_n) is cofinal in α, we have $h \in \bigcap_{\xi<\alpha} H_\xi$, and the conclusion follows.

(ii) Notice that we only need to check point (iv) with $\xi = \alpha$ from the definition of canonical approximations. (Note that point (ii) follows from (i) of the present lemma, and point (iii) follows from $(H_\xi, \tau_\xi)_{\xi<\alpha}$ being a canonical approximation for H.) Since countable intersections of comeager sets are comeager, it suffices to check that if $H \subseteq \bigcup_n F_n$, where for each n there exists $\xi_n < \alpha$ with F_n τ_{ξ_n}-closed, then $\bigcup_n F_n$ has τ_α-dense τ_α-interior. Since H is included in the union of these sets, it follows by our assumption on H' that $\bigcup_n F_n \cap H'$ is τ'-comeager. Thus,

$$\bigcup_n \tau'\text{-int}(F_n) \text{ is } \tau'\text{-dense in } H'. \tag{2.3}$$

Since each F_n is τ_{ξ_n}-closed for some $\xi_n < \beta$, it follows from Lemma 2.2 (applied to $S = H'$, $\sigma = \tau'$, $\gamma = \xi_n$, and $\xi = \xi_n + 1$), that

$$\bigcup_n \tau_{\xi_n+1}\text{-int}(F_n) \cap H_\alpha \supseteq \bigcup_n \tau'\text{-int}(F_n).$$

We also have

$$\bigcup_n \tau_\alpha\text{-int}(F_n) \supseteq \bigcup_n \tau_{\xi_n+1}\text{-int}(F_n) \cap H_\alpha.$$

as τ_α contains τ_{ξ_n+1} restricted to H_α. Thus, taking into account (2.3) and density of H' in H_α with respect to τ_α, we get that $\bigcup_n F_n$ has dense τ_α-interior and, therefore, is τ_α-comeager in H_α. Thus, indeed $(H_\xi, \tau_\xi)_{\xi<\alpha}$ extended by (H_α, τ_α) is a canonical approximation of H.

(iii) Clearly $H_\lambda \subseteq \bigcap_{\xi<\lambda} H_\xi$. Note further that $(H_\xi, \tau_\xi)_{\xi<\lambda}$ is a canonical approximation for H_λ. By (ii) above (applied to $H = H' = H_\lambda$), the sequence $(H_\gamma, \tau_\gamma)_{\gamma<\lambda}$ extended by $\bigcap_{\xi<\lambda} H_\xi$ taken with the Polish group topology on it given by (i) is a canonical approximation for H_λ as is $(H_\gamma, \tau_\gamma)_{\gamma<\lambda}$ extended by $(H_\lambda, \tau_\lambda)$. Using Lemma 2.1 we see that $\bigcap_{\xi<\lambda} H_\xi = H_\lambda$. □

2.2. **Two types of small sets.** Let G be a Polish group and let H be a subgroup of G. Two notions of small subsets of G will play a role in our proofs. First, define a set $F \subseteq G$ to be *l-small* if for any countable set $D \subseteq H$, DF does not contain H. (The letter "l" stands for "left.")

We will also need a notion of small sets which is stronger than l-smallness. Let H be an analytic subgroup of a Polish group G. We say that $F \subseteq G$ is *l-extra small* if there exists a continuous surjection $f : \mathbb{N}^\mathbb{N} \to H$ with the following property: for any countable $D \subseteq H$ and for any pruned trees $T_1, \ldots, T_n \subseteq \mathbb{N}^{<\mathbb{N}}$ there exists $Q \subseteq [T_1] \times \cdots \times [T_n]$ that is dense in some open non-empty subset of $[T_1] \times \cdots \times [T_n]$ and is such that

$$H \not\subseteq DF\{f(x_n)^{-1} \cdots f(x_1)^{-1} : (x_1, \ldots, x_n) \in Q\}. \tag{2.4}$$

In the above situation, we say that f witnesses that F is l-extra small. (In (2.4), we use $f(x)^{-1}$ rather than $f(x)$, which would have given an equivalent notion, since this convention makes certain formulas further on simpler.)

Note that l-extra small sets are l-small. As shown by Example 2.8 the two classes of sets are distinct in some situations.

2.3. **General results.** We state here the general results from which Theorems 1.1 and 1.2 will be deduced and which will be proved in Sections 3, 4, and 5. The following theorem gives a condition under which non-trivial canonical approximations, that is, of length 2 can be produced. Note that canonical approximations of length 1 always exist.

Theorem 2.4. *Let G be a Polish group, and let $H \subseteq G$ be an analytic subgroup. If H cannot be covered by countably many closed l-small sets, then there exists a canonical approximation for H of length 2, that is, there exists a Polishable subgroup H_1 of G with a Polish group topology τ_1 containing the topology H_1 inherits from G and such that*

(i) *$H \subseteq H_1$ and H is dense with respect to τ_1;*
(ii) *H_1 is $\mathbf{\Pi}^0_3$;*
(iii) *for any $\mathbf{\Pi}^0_3$ subset A of G with $H \subseteq A$, $A \cap H_1$ is comeager with respect to τ_1.*

The next result gives as a consequence bounds on the length of canonical approximations for a Borel subgroup H of a Polish group G in terms of the Borel class of H. This result is a generalization of [6, Theorem 3.1] to the case of not necessarily Polishable H, the case that is crucial for the considerations of the present paper. Note that each case listed in the conclusion of this theorem implies that if $(H_\xi, \tau_\xi)_{\xi<\beta}$ is a canonical approximation for H with $H_\xi \neq H_{\xi'}$ for distinct $\xi, \xi' < \beta$, then $\beta \leq \alpha$, which is all that we need in applications, and this estimate can, in fact, be gotten directly from Lemma 4.1. However, the additional information contained in Theorem 2.5 seems to be of some independent interest.

Theorem 2.5. *Let G be a Polish group, let H be a $\mathbf{\Pi}^0_{1+\alpha}$ subgroup of G that is not $\mathbf{\Sigma}^0_{1+\alpha}$, with $\alpha < \omega_1$, and let $(H_\xi, \tau_\xi)_{\xi<\beta}$ be a canonical approximation for H. Put*

$$H_\beta = \bigcap_{\xi<\beta} H_\xi \quad \text{and} \quad \beta_0 = \min\{\xi \leq \beta : H_\xi = H_\beta\}.$$

Then one of the following possibilities occurs

(i) *$H \neq H_\beta$ and $\beta \leq \alpha$, or*
(ii) *$H = H_{\beta_0}$, $\beta_0 + 1 = \alpha$, and β_0 is a successor, or*
(iii) *$H = H_{\beta_0}$, $\beta_0 = \alpha$, and β_0 is 0 or limit.*

Now we come to a theorem which gives a condition allowing us to embed $\mathbb{E}_1$ into the coset equivalence relation.

Theorem 2.6. *Let G be a Polish group, and let $H \subseteq G$ be an analytic subgroup. If H can be covered by countably many closed l-extra small sets, then $\mathbb{E}_1 \sqsubseteq_c E_{G/H}$.*

2.4. **Proofs of Theorems 1.1 and 1.2 from Theorems 2.4, 2.5, and 2.6.** We start with a lemma.

Lemma 2.7. *Let H be an analytic subgroup of a Polish group G.*

(i) *If H is Abelian, then l-small closed sets coincide with l-extra small closed sets.*

(ii) *If H is an increasing union of a sequence of Polishable subgroups of G, then l-small closed sets coincide with l-extra small closed sets.*

Proof. (i) is clear. We prove (ii). Let

$$H_0 < H_1 < \cdots < H_n < \cdots$$

be a sequence of Polishable subgroups of G whose union is H. Let f be a continuous surjection from $\mathbb{N}^{\mathbb{N}}$ onto H. Let $T_1, \ldots, T_k$ be pruned trees. By the Baire Category Theorem, there exists $n \in \mathbb{N}$, such that

$$\{(x_1, \ldots, x_k) \in [T_1] \times \cdots \times [T_k] : f(x_k)^{-1} \cdots f(x_1)^{-1} \in H_n\}$$

is non-meager in $[T_1] \times \cdots \times [T_k]$. Therefore, to show that a closed l-small sets F is l-extra small and that this is witnessed by f, it will suffice to show that if F is l-small and $n \in \mathbb{N}$, then for any countable sets $D \subseteq H$ and $Q \subseteq H_n$, we have $H \not\subseteq DFQ$.

Let τ_n be the Polish group topology on H_n witnessing its Polishability. Suppose towards a contradiction that there exists an n_0 and countable sets $D \subseteq H$ and $Q \subseteq H_{n_0}$ such that $DFQ \supseteq H = \bigcup_n H_n$. By the Baire category theorem, for each n there exist $d_n \in D$ and $q_n \in Q$ such that $d_n F q_n \cap H_n$ has non-empty τ_n-interior. Since $q_n \in H_{n_0}$ for all n, we have that for $n \geq n_0$, $q_n \in H_{n_0} \subseteq H_n$; thus, $d_n F \cap H_n$ has nonempty τ_n-interior for $n \geq n_0$. It follows that if $D' \subseteq H$ is a countable set such that $D' \cap H_n$ is τ_n-dense for each n, then $D' \bigcup_n d_n F$ contains H_n for $n \geq n_0$ and, therefore, it contains the whole H. Thus, F is not l-small, contradiction. □

Proof of Theorems 1.1 and 1.2. Let H be a subgroup of a Polish group G. Assume that H is Abelian and Borel or that H is an increasing union of a sequence of Polishable subgroups of G. In the second case, H is also Borel. There is $\alpha < \omega_1$ such that H is $\mathbf{\Pi}^0_{1+\alpha}$. By Theorem 2.5, there exists a maximal canonical approximation $(H_\xi, \tau_\xi)_{\xi<\beta}$ of H of countable length with $H_\xi \neq H_{\xi'}$ for distinct $\xi, \xi' < \beta$ (in fact, $\beta \leq \alpha$). If β is a successor, $\beta = \tilde{\beta}+1$, let $\tilde{H} = H_{\tilde{\beta}}$; if β is limit, let $\tilde{H} = \bigcap_{\xi<\beta} H_\xi$. By Lemma 2.3(i), in either case $\tilde{H}$ is a Polishable subgroup of G. Let $\tilde{\tau}$ be the Polish group topology on it making it Polishable. If $H = \tilde{H}$, H is Polishable and we are done. Assume therefore that H is a proper, and so $\tilde{\tau}$-meager, subgroup of the Polish group $\tilde{H}$.

We consider first the case when H cannot be covered by countably many closed l-small subsets of the Polish group $\tilde{H}$ (taken with the topology $\tilde{\tau}$). Apply Theorem 2.4 to obtain a Polishable subgroup H' of $\tilde{H}$ with a Polish group topology τ'. If β is a successor ordinal, we extend the sequence $(H_\xi, \tau_\xi)_{\xi<\beta}$ by (H', τ') to obtain a canonical approximation of H. This contradicts maximality of $(H_\xi, \tau_\xi)_{\xi<\beta}$ as it follows from Theorem 2.4(iii) and meagerness of H in $\tilde{\tau}$ that H' is a proper subgroup of $\tilde{H}$. If β is limit, the sequence $(H_\xi, \tau_\xi)_{\xi<\beta}$ extended by $(\tilde{H}, \tilde{\tau})$ (recall that $\tilde{H} = \bigcap_{\xi<\beta} H_\xi$) is also a canonical approximation of H by Lemma 2.3(ii), again contradicting maximality of $(H_\xi, \tau_\xi)_{\xi<\beta}$.

The considerations above show that H can be covered by countably many l-small closed subsets of $\tilde{H}$. By Lemma 2.7, H can be covered by countably many closed l-extra small sets in $\tilde{H}$. It follows from Theorem 1.2 that $\mathbb{E}_1 \sqsubseteq_c E_{\tilde{H}/H}$. Since the inclusion $\tilde{H} \subseteq G$ induces an embedding $E_{\tilde{H}/H} \sqsubseteq_c E_{G/H}$, we immediately get $\mathbb{E}_1 \sqsubseteq_c E_{G/H}$. □

2.5. **An example.** We will now give an example of a Polish group G and a Borel subgroup H of G such that Theorems 1.1 and 1.2 do not apply and yet Theorem 2.6 can be used to show that $\mathbb{E}_1 \sqsubseteq_c E_{G/H}$. It will also show that closed l-small sets can be distinct from closed l-extra small sets.

Example 2.8. *There is a Polish group G and a Borel subgroup H of G such that*

(i) *the assumptions of Theorems 1.1 and 1.2 fail,*
(ii) *the family of closed l-small sets is not closed under finite unions and there is a closed l-small set which is not l-extra small, and yet*
(iii) *Theorem 2.6 can be applied to show that $\mathbb{E}_1 \sqsubseteq_c E_{G/H}$.*

Let G be a Polish group that contains a perfect compact subset K of G consisting of free generators of the subgroup H generated by K. Such a G can be obtained by taking the completion of the Graev metric on the free group generated by the Cantor set $2^{\mathbb{N}}$ and taking $K = 2^{\mathbb{N}}$, see [8]. Alternatively, one can apply [1]. Note that H is $\mathbf{\Sigma}^0_2$. Clearly H is not Abelian. By [4] no uncountable subgroup of H is Polishable, in particular, H is not the union of an increasing sequence of Polishable subgroups. Thus, point (i) above is verified.

Fix $a \in K$. The set A_0 of all elements in H which when represented as an irreducible word in the alphabet $K \cup K^{-1}$ end in a or a^{-1} is σ-compact. Let $F_0 \subseteq H$ be compact and such that the union of countably many left translates of F_0 by elements of H is equal to A_0. Similarly, the set A_1 of all elements in H which when represented as an irreducible word in the alphabet $K \cup K^{-1}$ do not end in a or a^{-1} is σ-compact. Let F_1 be compact and such that the union of countably many left translates of F_1 by elements of H is equal to A_1. Then it is easy to see that both F_0 and F_1 are l-small (since $DA_0 \neq H$ and $DA_1 \neq H$ for any countable $D \subseteq H$). But countably many left translates of $F_0 \cup F_1$ cover $A_0 \cup A_1 = H$. It follows that $F_0 \cup F_1$ is not l-small.

Further, we claim that F_0 is not l-extra small. Otherwise, there is a continuous surjection f from $\mathbb{N}^{\mathbb{N}}$ onto H witnessing that F_0 is l-extra small. Clearly the same f witnesses that A_0 is l-extra small. Since $A_1 a \subseteq A_0$ the same is true about $A_1 a$ and hence by, Lemma 5.2(i), also about A_1. Therefore, by Lemma 5.2(ii), $A_0 \cup A_1$ is l-extra small, contradicting $H \subseteq A_0 \cup A_1$. Thus, (ii) is proved.

To see (iii), represent K as a union of compact subsets K_n, $n \in \mathbb{N}$, so that $K_n \subseteq K_{n+1}$ and $K_{n+1} \setminus K_n$ is uncountable. Consider the σ-compact subgroup of H generated by K_n, and let $F_{n,k}$, $k \in \mathbb{N}$, be compact subsets of this subgroup whose union covers it. Then since K_n is a subset of K_{n+1}, we have $\bigcup_{n,k} F_{n,k} = H$. Also for any countable subsets D, Q of H, the set $DF_{n,k}Q$ does not contain H. This is so since by countability of D and Q there exists $b \in K \setminus K_n$ which is not present in the irreducible form of any element of $D \cup Q$. Then $b \notin DF_{n,k}Q$. It follows that each $F_{n,k}$ is l-extra small as witnessed by any continuous surjection f from $\mathbb{N}^{\mathbb{N}}$ onto H. Thus, (iii) follows from Theorem 2.6.

3. Proof of the approximation theorem: Theorem 2.4

In this section, H is an analytic subgroup of a Polish group G.

If $T \subseteq \mathbb{N}^{<\mathbb{N}}$ is a tree and $t \in T$, by $[t]_T$ we denote the set of all $x \in [T]$ for which $t \subseteq x$. We start with a general observation.

Lemma 3.1. *Let $f : \mathbb{N}^{\mathbb{N}} \to H$ be a continuous surjection and let $\mathcal{F}$ be a family of closed subsets of G. Then no countable subfamily of it covers H if and only if there exists a nonempty pruned tree T such that for any $F \in \mathcal{F}$ and $t \in T$, $f([t]_T) \not\subseteq F$.*

Proof. If there exists a T as above, then for any $F \in \mathcal{F}$, $f^{-1}(F) \cap [T]$ is nowhere dense in $[T]$, therefore countably many sets from $\mathcal{F}$ cannot cover $f([T])$.

If countable subfamilies of $\mathcal{F}$ do not cover H, define a derivation on subtrees of $\mathbb{N}^{<\mathbb{N}}$ by letting

$$S' = \{t \in S : \forall F \in \mathcal{F}\ f([t]_S) \not\subseteq F\}$$

for a tree $S \subseteq \mathbb{N}^{<\mathbb{N}}$. Now we define a sequence of trees T_α for $\alpha < \omega_1$, by $T_0 = \mathbb{N}^{<\mathbb{N}}$, $T_{\alpha+1} = T'_\alpha$, and $T_\lambda = \bigcap_{\alpha<\lambda} T_\alpha$ for limit λ. Obviously, there exists a countable α_0 such that $T_{\alpha_0} = T_\gamma$ for all $\gamma \geq \alpha_0$ and with T_{α_0} nonempty. It is straightforward to check that $T = T_{\alpha_0}$ works. □

In the remainder of this section, we assume that countably many closed l-small sets cannot cover H. Note that, since l-small sets are closed under left translation by elements of H, this assumption immediately implies that the union of countably many closed l-small sets is l-small.

Under the above assumption, we will first prove some auxiliary lemmas. We introduce a couple more notions. $F \subseteq G$ is *r-small* if for any countable $D \subseteq H$, $FD \not\supseteq H$. A set will be called *l-large* if it is not l-small and it is *r-large* if it is not r-small.

Lemma 3.2. (i) *$\mathbf{\Sigma}^0_2$ l-large sets coincide with $\mathbf{\Sigma}^0_2$ r-large sets.*

(ii) *There exists a countable set $Q \subseteq H$ such that if F is $\mathbf{\Sigma}^0_2$ l-large, then $QF \supseteq H$.*

(iii) *There exists a sequence F_n, $n \in \mathbb{N}$, of closed l-large sets such that if F is $\mathbf{\Sigma}^0_2$ l-large, then $F_n \subseteq F$ for some n.*

(iv) *If F_1, F_2 are two $\mathbf{\Sigma}^0_2$ l-large sets, then there exist $h_1, h_2 \in H$ such that $F_1 \cap h_1F_2$ and $F_1 \cap F_2h_2$ are l-large.*

Proof. Since H is analytic and cannot be covered by countably many $\mathbf{\Sigma}^0_2$ l-small sets, by Lemma 3.1, there exists a nonempty pruned tree T such that if F is $\mathbf{\Sigma}^0_2$ and l-small, then $f^{-1}(F) \cap [T]$ is meager in $[T]$. Put

$$P = f([T]).$$

Let t_n, $n \in \mathbb{N}$, list all the elements of T. Define

$$F'_n = \overline{f([t_n]_T)}.$$

Clearly each F'_n is l-large. Thus, we can find countable sets $Q_n \subseteq H$ such that $Q_nF'_n \supseteq H$. Define

$$Q = \bigcup_n Q_n.$$

We show now that $\mathbf{\Sigma}^0_2$ r-large sets are l-large and, moreover, that there exists one countable subset of H, namely the set Q defined above, such that if F is $\mathbf{\Sigma}^0_2$ r-large, then $QF \supseteq H$. Let F be $\mathbf{\Sigma}^0_2$ r-large. Then for some countable $Q' \subseteq H$, $FQ' \supseteq H \supseteq P$. By the Baire Category Theorem there exist $h \in Q'$ and $n \in \mathbb{N}$ such that $F'_n \subseteq Fh$. Since F'_n is l-large, F is l-large. Moreover, it follows that $H \subseteq QF'_n \subseteq QFh$, whence $H \subseteq QF$.

Note that if F is $\mathbf{\Sigma}^0_2$ r-small, then F^{-1} is $\mathbf{\Sigma}^0_2$ l-small. Thus, if $H \subseteq \bigcup_n F^n$ with each F^n $\mathbf{\Sigma}^0_2$ r-small, then $H = H^{-1} \subseteq \bigcup_n (F^n)^{-1}$, that is, H could be covered by countably many l-small sets. Therefore, it follows from our assumption that H cannot be covered by countably many $\mathbf{\Sigma}^0_2$ r-small sets. Thus, by an argument as in the previous paragraph, we get that $\mathbf{\Sigma}^0_2$ l-large sets are r-large. Combining the two conclusion we get (i). Moreover, by what was proved above about Q, we get (ii) as well.

Let $Q = \{h_k : \ k \in \mathbb{N}\}$. Define F_m, $m \in \mathbb{N}$, to be an enumeration of the family $\{h_k^{-1} F'_n : \ k, n \in \mathbb{N}\}$. Clearly all the sets F_m are closed l-large. If F is $\mathbf{\Sigma}^0_2$ l-large, then by (ii), proved above, $P \subseteq H \subseteq QF$. So, by the Baire Category Theorem, for some k and n, $F'_n \subseteq h_k F$, that is $h_k^{-1} F'_n \subseteq F$. Thus (iii) follows.

To see (iv), let $D \subseteq H$ be countable and such that DF_2 covers H. It follows that $F_1 \cap DF_2$ is l-large since it contains $F_1 \cap H$. Thus, $F_1 \cap h_1 F_2$ is l-large for some $h_1 \in D$ and the first part of (iv) is proved. Similarly, one shows that if F_1 and F_2 are $\mathbf{\Sigma}^0_2$ r-large, then there exists $h_2 \in H$ with $F_1 \cap F_2 h_2$ r-large. But since $\mathbf{\Sigma}^0_2$ r-large sets are precisely $\mathbf{\Sigma}^0_2$ l-large sets, the second part of point (iv) follows. $\square$

In view of Lemma 3.2(i), we will drop the "l" and the "r" from "l-large" and "r-large" if the set in question is $\mathbf{\Sigma}^0_2$ and we will simply call such sets large. Similarly for small sets.

Lemma 3.3. *For any closed large set F there exist $h \in H$ and a closed large set F_1 such that $hF_1F_1^{-1} \subseteq F$.*

Proof. For $A \subseteq G \times G$ and $g \in G$, let $A^g = \{g' \in G : (g', g) \in A\}$.

Now, fix a closed large set F. We will first show that there exist closed large sets F', F'' with $F'F'' \subseteq F$. Let Q be a countable set as in Lemma 3.2(ii). Put $Q = \{h_n : \ n \in \mathbb{N}\}$. Let

$$A_n = \{(g_1, g_2) \in G \times G : \ g_1 g_2 \in h_n F\}.$$

Note that A_n is closed and that for any $h \in H$

$$\bigcup_n (A_n)^h = QFh^{-1} \supseteq Hh^{-1} = H.$$

Thus, for any $h \in H$ there is an n with $(A_n)^h$ large. By Lemma 3.2(iii), we can fix a sequence F_m, $m \in \mathbb{N}$, of closed large sets such that any closed large set contains one of the F_m's. Let

$$B_{m,n} = \{g \in G : \ F_m \subseteq (A_n)^g\}.$$

Each $B_{m,n}$ is closed and, by what was noticed above, $H \subseteq \bigcup_{m,n} B_{m,n}$. Thus for some m_0, n_0, B_{m_0,n_0} is large. It follows that $F_{m_0} \times B_{m_0,n_0} \subseteq A_{n_0}$ whence $(h_{n_0}^{-1} F_{m_0}) B_{m_0,n_0} \subseteq F$. Let $F' = h_{n_0}^{-1} F_{m_0}$ and $F'' = B_{m_0,n_0}$.

Since $(F'')^{-1}$ and F' are large, by Lemma 3.2(iv), there exists $h \in H$ with $(F'')^{-1} \cap hF'$ large and certainly closed. Let $F_1 = (F'')^{-1} \cap hF'$. Then $F_1^{-1} \subseteq F''$ and $h^{-1}F_1 \subseteq F'$, so $h^{-1}F_1F_1^{-1} \subseteq F$. □

We are ready now to prove Theorem 2.4.

Proof of Theorem 2.4. Define

$$H_1 = \{g \in G : \ \forall F \text{ closed large } \exists h_1, h_2 \in H \ g \in h_1F \text{ and } g \in Fh_2\}.$$

Claim 1. H_1 is a $\mathbf{\Pi}^0_3$ subgroup of G with $H \subseteq H_1$.

Proof of Claim 1. The inclusion $H \subseteq H_1$ is obvious. We show now that H_1 is a group. Let $g_1, g_2 \in H_1$. We need to see that $g_1 g_2^{-1} \in H_1$. Let F be closed large. By symmetry, it is enough to prove that $g_1 g_2^{-1} \in hF$ for some $h \in H$. Let F_1 be closed large and such that $F_1 F_1^{-1} \subseteq h_1 F$ for some $h_1 \in H$. This can be done by Lemma 3.3. For some $h \in H$, $g_2 \in F_1 h$, so $g_2^{-1} \in h^{-1} F_1^{-1}$. The set $F_1 h$ is large so for some $h' \in H$, $g_1 \in h' F_1 h$. Thus

$$g_1 g_2^{-1} \in h' F_1 h h^{-1} F_1^{-1} = h' F_1 F_1^{-1} \subseteq h' h_1 F$$

and $h' h_1 \in H$.

It remains to see that H_1 is $\mathbf{\Pi}^0_3$. Let Q be a countable subset of H as in Lemma 3.2(ii). We now observe that for any F closed large, $H_1 \subseteq QF \cap FQ^{-1}$. To see this, let F be closed large and let $g \in H_1$. By Lemma 3.3, find F_1, F_2 closed large with $F_1F_2 \subseteq F$. Then $g \in hF_2$ for some $h \in H$. From the definition of Q, $h \in h_1F_1$ for some $h_1 \in Q$. Hence $g \in h_1F_1F_2 \subseteq h_1F$ and $H_1 \subseteq QF$ follows. Since if F is closed large, F^{-1} is closed large, so by the argument above, $H_1 \subseteq QF^{-1}$, whence $H_1 \subseteq FQ^{-1}$.

It follows from the above observation that if F_n, $n \in \mathbb{N}$, are closed large sets such that any closed large set contains one of the F_n's as in Lemma 3.2(iii), then

$$H_1 = \bigcap_n (QF_n \cap F_n Q^{-1}).$$

But the right hand side of the above equality is clearly $\mathbf{\Pi}^0_3$, and Claim 1 is proved.

We will now construct a group topology τ_1 on H_1 that witness Polishability of H_1. Let F_n, $n \in \mathbb{N}$, be a sequence of closed large sets such that for any closed large set F, $F_n \subseteq F$ for some n. By induction on n we construct a sequence of closed large sets D_n so that

(1) $\forall k \leq n \exists h_1, h_2 \in H \; (h_1 D_n D_n^{-1} \subseteq F_k$ and $D_n D_n^{-1} h_2 \subseteq F_k)$;
(2) $\forall k < n \exists h \in H \; (D_n D_n^{-1} h \subseteq D_k)$.

To do this, assume $n = 0$, or $n > 0$ and $D_0, \ldots, D_{n-1}$ are defined, and consider the family $\{F_k : \; k \leq n\} \cup \{D_k : \; k < n\}$. It consists of $2n+1$ closed large sets, so by Lemma 3.3 there exist closed large sets D^i, $i \leq 3n+1$, such that

$\forall i \leq n \exists h \in H \; hD^i(D^i)^{-1} \subseteq F_i$;
$\forall n+1 \leq i \leq 2n+1 \exists h \in H \; D^i(D^i)^{-1}h \subseteq F_{i-n-1}$;
$\forall 2n+2 \leq i \leq 3n+1 \exists h \in H \; D^i(D^i)^{-1}h \subseteq D_{i-2n-2}$.

To find a closed large set D_n fulfilling (1) and (2), it suffices now to produce a closed large set D_n such that for any $i \leq 3n+1$, $D_n h_i \subseteq D^i$ for some $h_i \in H$. To construct such a D_n, we repeatedly apply Lemma 3.2(iv) to find $h_i \in H$, $i \leq 3n+1$, with $\bigcap_{i \leq 3n+1} D^i h_i^{-1}$ large and we let $D_n = \bigcap_{i \leq 3n+1} D^i h_i^{-1}$.

Having defined D_n for $n \in \mathbb{N}$, let

$$V_n = \overline{D_n D_n^{-1}}.$$

Then

(α) V_n is closed large;
(β) $\forall F$ closed large $\exists n \; \exists h_1, h_2 \in H \; h_1 V_n \subseteq F$ and $V_n h_2 \subseteq F$;
(γ) if $U \ni 1$ is open in G, then $V_n \subseteq U$ for some n;
(δ) V_n is symmetric ($V_n^{-1} = V_n$) and $1 \in V_n$;
(ϵ) $V_{n+1}^2 \subseteq V_n$.
(ζ) $\forall n \forall g \in H_1 \exists k \; V_n \supseteq g V_k g^{-1}$.

Here (α) and (δ) are obvious, (β) follows from (1) and the choice of the F_n's, (ϵ) follows from (2), and (γ) is a consequence of (1) and the fact that

$1 \in V_n$ and that if U is an open neighborhood of 1 in G, then $WW^{-1} \subseteq U$ for some open nonempty set W and $\overline{W}$ is large. To see (ζ), fix n and $g \in H_1$. By (α), we can find $h \in H$ such that $g \in V_{n+3}h^{-1}$. Then $h \in g^{-1}V_{n+3}$. Since $h \in H$, $hV_{n+3}h^{-1}$ is large and closed, so by (β) for some $h_1 \in H$ and some $k \in \mathbb{N}$, $hV_{n+3}h^{-1} \supseteq V_k h_1$. Thus by ($\delta$) and ($\epsilon$)

$$g^{-1}V_{n+1}g \supseteq g^{-1}V_{n+3}^3 g \supseteq hV_{n+3}h^{-1} \supseteq V_k h_1.$$

It follows that

$$g^{-1}V_n g \supseteq g^{-1}V_{n+1}gg^{-1}V_{n+1}g \supseteq V_k h_1 (V_k h_1)^{-1} = V_k^2 \supseteq V_k.$$

Our aim is to apply [18, Theorem 10], which gives conditions on a family of subsets of an abstract group ensuring the existence of a group topology for which the given family of subsets becomes an open basis at 1. To apply this theorem, we need to modify the sets V_n and define

$$U_n = \{h \in V_n \cap H_1 : \exists k \; V_k h \subseteq V_n\}. \tag{3.1}$$

We claim that the sets U_n fulfill the following conditions

(a) $V_{n+1} \cap H_1 \subseteq U_n$;
(b) $1 \in U_n \subseteq V_n$;
(c) $U_{n+1}U_{n+1}^{-1} \subseteq U_n$;
(d) for any $h \in U_n$ there exists k with $U_k h \subseteq U_n$;
(e) for any $g \in H_1$ and any n there exists k with $gU_k g^{-1} \subseteq U_n$.

Point (a) follows from (ϵ) and (δ); (b) is obvious; (c) is seen by the following computation, which uses (a), (b), (δ), and (ϵ):

$$U_{n+1}U_{n+1}^{-1} \subseteq (V_{n+1} \cap H_1)^2 \subseteq V_n \cap H_1 \subseteq U_n.$$

To see (d), let $h \in U_n$. The definition of U_n allows us to fix k with $V_k h \subseteq V_n$. We claim that $U_k h \subseteq U_n$. Let $h' \in U_k$. Fix m with $V_m h' \subseteq V_k$. Then,

$$V_m h' h \subseteq V_k h \subseteq V_n,$$

which puts $h'h$ in U_n as required. Finally, we show (e). Let n and $g \in H_1$ be given. By (ζ), we can find k with $gV_k g^{-1} \subseteq V_n$. We claim that k works, that is, that $gU_k g^{-1} \subseteq U_n$. To see this, let $h \in U_k$. Fix m_1 with $V_{m_1} h \subseteq V_k$, and by (ζ) fix m_2 with $V_{m_2} \subseteq gV_{m_1}g^{-1}$. Putting all this together we get

$$V_{m_2} ghg^{-1} \subseteq gV_{m_1}g^{-1}ghg^{-1} \subseteq gV_{m_1}hg^{-1} \subseteq gV_k g^{-1} \subseteq V_n.$$

Thus, $ghg^{-1} \in U_n$ as required.

By (b)–(e) and (γ) above, we can apply [18, Theorem 10] to obtain a group topology τ_1 on H_1 such that the sequence (U_n) forms the neighborhood basis at 1 consisting of open sets. In what follows, we will use conditions (a)–(e) and (α)–(γ).

Claim 2. τ_1 is a metrizable separable topology on H_1 that contains the topology inherited by H_1 from the inclusion $H_1 \subseteq G$. Furthermore, H is τ_1-dense in H_1.

Proof of Claim 2. Since τ_1 has a countable basis (U_n) at 1, by [9, Theorem 8.3], it is metrizable.

Let $Q \subseteq H$ be as in Lemma 3.2(ii). We claim that Q is τ_1-dense in H_1. Let $g \in H_1$ and let $n \in \mathbb{N}$. We need to find $h \in Q$ with $h^{-1}g \in U_n$. So, by (a) it is enough to find $h \in Q$ with $g \in hV_{n+1}$. Since, by (α), V_{n+2} is closed large, by definition of H_1, we can find $h_1 \in H$ with $g \in h_1V_{n+2}$. But $h_1 \in hV_{n+2}$ for some $h \in Q$ so $g \in hV_{n+2}^2 \subseteq hV_{n+1}$. Thus, we proved that τ_1 is separable and that H is τ_1-dense in H_1.

To see that τ_1 is stronger than the topology inherited from the inclusion $H_1 \subseteq G$, note that both these topologies are group topologies, so it is enough to check that any $U \ni 1$ open in G contains U_n for some n. This follows from (γ) and (b). This finishes the proof of Claim 2.

Recall that the *strong Choquet game* on a topological space is an infinite game, played by two players taking turns. So moves are indexed by natural numbers and Player I plays the even numbered moves (the numbering starts at 0) while Player II plays the odd numbered moves. Player I starts with playing an open set and a point in it. After Player I played his move, which consists of an open set W and a point $x \in W$, Player II responds by playing an open set V with $x \in V \subseteq W$; after Player II played his move, which consists of an open set V, Player I responds by playing an open set $W \subseteq V$ and a point $x \in W$. Player II is declared a winner of a run of the game if the intersection of all open sets played in the run by either player, equivalently both players, is non-empty. A topological space is called *strong Choquet* if there is a winning strategy for Player II in the strong Choquet game played on this space. For more information on the strong Choquet game see [11].

Claim 3. τ_1 is a strong Choquet topology.

Proof of Claim 3. We need to describe a strategy for player II in the strong Choquet game. Fix a complete metric d_1 on G compatible with the Polish group topology on G. We will define the $2k+1$-st move played by II. Let I play the $2k$-th move consisting of W_k and g with $W_k \subseteq H_1$ τ_1-open and $g \in W_k$. Let $W \subseteq G$ be open in G and such that $g \in W$ and

d-diam$(W) \leq 1/(k+1)$. By Claim 2, $W \cap W_k$ is a τ_1-open set containing g. Since H is τ_1-dense in H_1 and since τ_1 is a group topology, we have

$$\exists V \text{ a } \tau_1\text{-open neighborhood of } 1 \exists h \in H \ (hV \subseteq W \cap W_k \text{ and } hV \cap Vh \ni g).$$

Thus, using additionally (a), we can find $h_k \in H$ and $n_k > k$ such that

$$g \in h_k U_{n_k} \cap U_{n_k} h_k \subseteq W \cap W_k \tag{3.2}$$

and

$$h_k V_{n_k} \cap H_1 \subseteq W \cap W_k. \tag{3.3}$$

We let II play $O_k = h_k U_{n_k} \cap U_{n_k} h_k$. By (3.2), it is a legal move.

Note that d-diam$(O_k) \leq 1/(k+1)$ since $O_k \subseteq W$. Also by (b), $\overline{O_k} \subseteq h_k V_{n_k} \cap V_{n_k} h_k$, where the closure in $\overline{O_k}$ is taken in the Polish topology on G. Since d is complete, there exists $g_\infty \in G$ with $g_\infty \in \bigcap_k \overline{O_k}$. It follows that $g_\infty \in \bigcap_k h_k V_{n_k} \cap V_{n_k} h_k$, with $h_k \in H$. Thus, by (β), by $V_{n+1} \subseteq V_n$, which follows from (δ) and (ϵ), and by the definition of H_1, we get $g_\infty \in H_1$. This means that, for any k, $g_\infty \in h_k V_{n_k} \cap H_1$, which, in turn, by (3.3) gives $g_\infty \in \bigcap_k W_k$, proving that II wins the game. This establishes the claim.

It follows from Claims 2 and 3, by Choquet's theorem [11], that τ_1 a Polish topology. Thus, using again Claim 2, we see that H_1 is a Polishable subgroup of G which, by Claim 1, is $\mathbf{\Pi}^0_3$. Therefore, it remains only to prove point (iii) of Theorem 2.4.

Let A be as in the assumption of (iii). Since the intersection of countably many comeager sets is comeager, without loss of generality we can assume that A is $\mathbf{\Sigma}^0_2$, so $A = \bigcup_i L_i$ with each L_i, $i \in \mathbb{N}$, closed. Let V be a non-empty τ_1-open set. By (a) there exists $h \in H_1$ and n with

$$hV_n \cap H_1 \subseteq V. \tag{3.4}$$

Since hV_n is large and $H \subseteq \bigcup_i L_i$, we see that $hV_n \cap \bigcup_i L_i$ is large. Since hV_n and each L_i are closed and countable unions of closed small sets are small, there exists i_0 with $hV_n \cap L_{i_0}$ large. Thus, by definition of H_1, countably many left translates of $hV_n \cap L_{i_0}$ and, therefore, also countably many left translates of $hV_n \cap L_{i_0} \cap H_1$, by elements of H cover H_1. It follows by the Baire category theorem that $hV_n \cap L_{i_0} \cap H_1$ is non-meager with respect to τ_1. By (3.4), this implies that $L_{i_0} \cap V$ is τ_1-non-meager. Since V was an arbitrary non-empty τ_1-open set, it follows that $A \cap H_1 = \bigcup_i L_i \cap H_1$ is τ_1-comeager, and (iii) is proved. $\square$

4. Proof of the stabilization theorem: Theorem 2.5

Lemmas 2.2 and 2.3 proved in Subsection 2.1 will play an important role in the arguments of this section. We prove now a lemma on termination of canonical approximations.

Lemma 4.1. *Let H be a $\mathbf{\Pi}^0_{1+\alpha}$ subgroup of a Polish group G, and let $(H_\xi, \tau_\xi)_{\xi<\beta}$ be a canonical approximation for H. If $\alpha < \beta$, then $H = H_\alpha$.*

Proof. Assume that $\alpha < \beta$. Here is the basic definition for this proof. Call a set $A \subseteq G$ *slight* if there are sets F_n, $n \in \mathbb{N}$, such that $A \cap H_\alpha \subseteq \bigcup_n F_n$, τ_α-int$(F_n) = \emptyset$, and for each n there exists $\xi < \alpha$ depending on n such that $F_n \cap H_\xi$ is τ_ξ-closed. Note that slight sets are closed under taking subsets and countable unions.

We will show the following two claims.

Claim 1.
- (i) If U is non-empty and τ_α-open, then $U \cap H$ is not slight.
- (ii) If $B \subseteq H_\alpha$ is τ_α-non-meager, then B is not slight.
- (iii) Let $\gamma < \xi \leq \alpha$. Let $A, F_1, F_2 \subseteq G$, with $F_1 \cap H_\gamma$ τ_γ-closed and $F_2 \cap H_\xi$ τ_ξ-closed, be such that $A \cap F_1$ is slight and $A \cap V$ is not slight for any τ_ξ-open set V with $V \cap F_2 \neq \emptyset$. Then the τ_α-interior of $F_1 \cap F_2$ is empty.

Claim 2. If $A \subseteq G$ is $\mathbf{\Pi}^0_{1+\xi}$ for $\xi \leq \alpha$, then there exists $F \subseteq G$ such that
- (i) $F \cap H_\xi$ is τ_ξ-closed;
- (ii) $A \setminus F$ is slight;
- (iii) if $\xi < \alpha$, then $F \setminus A$ is slight; if $\xi = \alpha$, then $(F \setminus A) \cap H_\alpha$ is τ_α-meager.

Before proving the claims, we show how to deduce the theorem from them. Since H is $\mathbf{\Pi}^0_{1+\alpha}$, by Claim 2 with $\xi = \alpha$, there exists F such that $F \cap H_\alpha$ is τ_α-closed, $H \setminus F$ is slight, and $F \setminus H$ is τ_α-meager. Since $H_\alpha \setminus F$ is τ_α-open, $H \subseteq H_\alpha$, and $H \setminus F$ is slight, we see by Claim 1(i) that $H \subseteq F$. This implies, since H is τ_α-dense and $F \cap H_\alpha$ is τ_α-closed, that $H_\alpha \subseteq F$. This, in turn, yields

$$H_\alpha \setminus H \subseteq F \setminus H.$$

Since the set on the right side of the inclusion when intersected with H_α is τ_α-meager, so is the set on the left side. Since H is a subgroup of H_α, we get $H = H_\alpha$.

Proof of Claim 1. (i) Otherwise, $U \cap H \subseteq \bigcup_n F_n$ with $F_n \cap H_\alpha$ nowhere dense in τ_α and $F_n \cap H_{\xi_n}$ τ_{ξ_n}-closed for some $\xi_n < \alpha$. For some countable

$D \subseteq H$ we get

$$H = D(U \cap H) \subseteq D \bigcup_n F_n$$

and $dF_n \cap H_{\xi_n}$ is closed in τ_{ξ_n} for any $d \in D$; thus, by definition of canonical approximation,

$$D \bigcup_n F_n \cap H_\alpha$$

is comeager in τ_α contradicting nowhere density of each $F_n \cap H_\alpha$.

(ii) Again, assume towards a contradiction that $B \subseteq \bigcup_n F_n$ with F_n not containing a non-empty τ_α-open set and with $F_n \cap H_{\xi_n}$ τ_{ξ_n}-closed for some $\xi_n < \alpha$. For some n_0, $F_{n_0} \cap H_\alpha$ is τ_α-non-meager. Therefore, $F_{n_0} \cap H_\alpha$ being τ_α-closed contains a τ_α-open set, contradiction.

(iii) Assume towards a contradiction that there exists a non-empty τ_α-open set U with $U \subseteq F_1 \cap F_2$. By Lemma 2.2 with $S = H_\alpha$, since $\gamma < \xi \leq \alpha$ and $F_1 \cap H_\gamma$ is τ_γ-closed, there exists a τ_ξ-open set V with $U \subseteq V \subseteq F_1$. By our assumption on F_1, $A \cap V$ is slight, which, by our assumption on F_2, gives $V \cap F_2 = \emptyset$. This, contradicts the inclusion $U \subseteq V \cap F_2$, and the claim is proved.

Proof of Claim 2. For $A \subseteq G$ and $\xi \leq \alpha$ define

$$c_\xi(A) = G \setminus \bigcup \{U : A \cap U \text{ is slight and } U \text{ is } \tau_\xi\text{-open}\}.$$

We claim that $F = c_\xi(A)$ fulfils the conclusion of the claim.

Clearly $c_\xi(A) \cap H_\xi$ is τ_ξ-closed, so (i) holds. Note that since slight sets are closed under countable unions and τ_ξ is second countable, the set $A \setminus c_\xi(A)$ is always slight, and (ii) follows.

It remains to show that for a $\mathbf{\Pi}^0_{1+\xi}$ set A, $c_\xi(A) \setminus A$ is slight when $\xi < \alpha$ and $(c_\xi(A) \setminus A) \cap H_\alpha$ is τ_α-meager when $\xi = \alpha$. We accomplish it by induction on ξ. If $\xi = 0$, then A is closed in G so we have $c_0(A) \subseteq A$ and the conclusion follows immediately. Assume we have proved it for all $\gamma < \xi$. Let A be $\mathbf{\Pi}^0_{1+\xi}$. This means that we can find B_n in $\mathbf{\Pi}^0_{1+\gamma_n}$ for some $\gamma_n < \xi$ with $G \setminus A = \bigcup_n B_n$. Note that

$$c_\xi(A) \setminus A = c_\xi(A) \cap \bigcup_n B_n \subseteq \bigcup_n (c_\xi(A) \cap c_{\gamma_n}(B_n)) \cup (B_n \setminus c_{\gamma_n}(B_n)).$$

By (ii) proved above, $B_n \setminus c_{\gamma_n}(B_n)$ is slight. It also means, by Claim 1(ii), that the intersection of this set with H_α is τ_α-meager. Therefore, it will suffice to appropriately estimate the size of the sets $c_\xi(A) \cap c_{\gamma_n}(B_n)$, $n \in \mathbb{N}$, by proving that they are slight if $\xi < \alpha$ and that their intersection with H_α is τ_α-meager if $\xi = \alpha$.

We check that $A \cap c_{\gamma_n}(B_n)$ is slight and that $A \cap V$ is not slight for any τ_ξ-open set V with $V \cap c_\xi(A) \neq \emptyset$. The second condition follows directly from the definition of $c_\xi(A)$. To see the first one, note that

$$A \cap c_{\gamma_n}(B_n) \subseteq c_{\gamma_n}(B_n) \setminus B_n,$$

and by our inductive assumption $c_{\gamma_n}(B_n) \setminus B_n$ is slight. It follows now from Claim 1(iii) with $F_1 = c_{\gamma_n}(B_n)$ and $F_2 = c_\xi(A)$ that $c_\xi(A) \cap c_{\gamma_n}(B_n)$ has empty τ_α-interior. Since the intersection of this set with H_ξ is τ_ξ-closed (as $\gamma_n < \xi$), we see, from the very definition of slightness, that this set is slight if $\xi < \alpha$ and, from the definition of meagerness, that its intersection with H_α is τ_α-meager if $\xi = \alpha$.

This finishes the proof of the claim and the theorem. □

Proof of Theorem 2.5. Assume first that $H \neq H_\beta$. Then, by Lemma 4.1, $\beta \leq \alpha$, and point (i) in Theorem 2.5 holds.

Assume now that $H = H_\beta$. Then by Lemma 2.3(i), H is Polishable. Let τ_β be the Polish group topology on H_β witnessing its Polishability. If $\beta_0 < \beta$, then $(H_\xi, \tau_\xi)_{\xi \leq \beta_0}$ is a canonical approximation for H with $H = H_{\beta_0}$. If $\beta_0 = \beta$, then the same is true by Lemma 2.3(ii). Thus, in either case we have a canonical approximation $(H_\xi, \tau_\xi)_{\xi \leq \beta_0}$ for H such that $H_{\beta_0} = H$ and $H_\xi \neq H$ for $\xi < \beta_0$. Now [6, Theorem 3.1] states that in this situation if β_0 is a successor ordinal, then $\alpha = \beta_0 + 1$, and if β_0 is 0 or a limit ordinal, then $\alpha = \beta_0$. Thus, points (ii) or (iii) hold. □

Alain Louveau points out that one can avoid using [6, Theorem 3.1] in the argument above by appropriately strengthening Lemma 4.1 in the case of Polishable H. This is accomplished by a careful modification of the notion of slightness from the proof of this lemma. (This strengthening of Lemma 4.1 also follows from [6, Theorem 3.1], whose proof however uses different methods.) Louveau also points out that there exist interesting parallels between considerations of this section and those of [14, Section 1].

5. Proof of the embedding theorem: Theorem 2.6

Basic notation related to trees is recalled at the beginning of Sections 2 and 3. For $t \in \mathbb{N}^{<\mathbb{N}}$, let $|t|$ be the length of t, that is, the unique $n \in \mathbb{N}$ with $t \in \mathbb{N}^n$. If t has length n and $l \leq n$ or if $x \in \mathbb{N}^\mathbb{N}$ and $l \in \mathbb{N}$, then by $t \restriction l$ or $x \restriction l$ we denote the unique element of $\mathbb{N}^l$ extended by t or x, respectively. We will use the quantifiers $\exists^*$ and $\forall^*$ which are defined as follows. For a Polish space X we write $\exists^* x \in X\ \phi(x)$ if the condition $\phi(x)$ holds for a non meager set of $x \in X$. Similarly, if $\phi(x)$ holds for a comeager set of $x \in X$, we write $\forall^* x \in X\ \phi(x)$.

Let H be an analytic subgroup of a Polish group G. Fix a continuous surjection $f : \mathbb{N}^{\mathbb{N}} \to H$. For the sake of simplicity of notation, we will introduce some auxiliary definitions. A tree T on $\mathbb{N}^m$ is called a *product tree* if for some trees $T_1, \ldots, T_m$ on $\mathbb{N}$,

$$\begin{aligned} T &= T_1 \otimes \cdots \otimes T_m \\ &= \{(t_1, \ldots, t_m) : |t_i| = |t_j| \text{ and } t_i \in T_i \text{ for } 1 \leq i, j \leq m\}. \end{aligned}$$

If a tree T on $\mathbb{N}^m$ is a product tree, a function $g : [T] \to H$ is called a *product function* if for some $q_1, \ldots, q_m \in H$ we have for $x = (x_1, \ldots, x_m) \in [T]$

$$g(x) = q_1 \cdot f(x_1) \cdots q_m \cdot f(x_m).$$

We also adopt the following convention. If a set A is l-extra small and a continuous surjection f from $\mathbb{N}^{\mathbb{N}}$ to H witnesses it, then we call A *f-l-extra small.*

Lemma 5.1. *Let $f : \mathbb{N}^{\mathbb{N}} \to H$ be a continuous surjection. Let F be a closed subset of G. Then F is f-l-extra small if and only if for any product tree T and any product function g there exists a pruned tree S on $\mathbb{N}$ and $t \in T$ such that*

$$\forall h \in H \forall^* x \in [t]_T \forall^* y \in [S]\ f(y)g(x) \notin hF.$$

Proof. ($\Rightarrow$) Note that it will suffice to prove the right side for an arbitrary product tree T on $\mathbb{N}^m$ but only for a product function $g : [T] \to H$ of the form $g(x) = f(x_1) \cdots f(x_m)$ where $x = (x_1, \ldots, x_m)$. Indeed, to get the same statement for an arbitrary product function $g'(x) = q_1 f(x_1) \cdots q_k f(x_k)$ with $q_1, \ldots, q_k \in H$ defined on a product tree $T' = T'_1 \otimes \cdots \otimes T'_k$, it is enough to apply it to the product tree

$$T = S_1 \otimes T'_1 \otimes \cdots S_k \otimes T'_k,$$

with $S_i = \{y_i \restriction n : n \in \mathbb{N}\}$, where $y_i \in \mathbb{N}^{\mathbb{N}}$ is chosen so that $f(y_i) = q_i$, and to the product function $g(x) = f(x_1) \cdots f(x_{2k})$.

Assume the condition from the lemma fails. This means, using the fact that F is closed, that we can fix a product tree T and a product function g so that for any $t \in T$ and for any pruned tree S on $\mathbb{N}$ there exists $s \in S$ such that

$$\exists h \in H \exists^* x \in [t]_T\ f([s]_S) \subseteq hFg(x)^{-1}$$

Let $t \in T$ be given. The above property makes it possible, by a transfinite exhaustion argument, to find sets A^t_n, $n \in \mathbb{N}$, such that

$$H = \bigcup_n A^t_n \text{ and } \forall n \exists h^t_n \in H \exists^* x \in [t]_T \; A^t_n \subseteq h^t_n F g(x)^{-1}.$$

Define $D = \{h^t_n : n \in \mathbb{N}, t \in T\}$. Let now $Q \subseteq [T]$ be somewhere dense. Say, it is dense in $[t_0]_T$ for some $t_0 \in T$. Obviously, for each n

$$\{x \in [t_0]_T : A^{t_0}_n \subseteq h^{t_0}_n F g(x)^{-1}\} \subseteq \{x \in [t_0]_T : A^{t_0}_n \subseteq DFg(x)^{-1}\}.$$

Note now that the first of these two sets is non-meager and obviously closed, so it contains a nonempty open subset of $[t_0]_T$. It follows that for each n there exists an $x_n \in Q$ such that $A^{t_0}_n \subseteq DFg(x_n)^{-1}$. This implies

$$H = \bigcup_n A^{t_0}_n \subseteq DFg(Q)^{-1}$$

which means that F is not f-l-extra small since g is of the form $g(x) = f(x_1) \cdots f(x_m)$.

($\Leftarrow$) Assume now that F is not f-l-extra small. Fix a countable set $D \subseteq H$, a product tree T and a product function g so that for any $Q \subseteq [T]$ which is somewhere dense we have $H \subseteq DFg(Q)^{-1}$. Let now $t \in T$ and a pruned tree S on $\mathbb{N}$ be given. It will suffice to see that

$$\exists^* x \in [t]_T \; (\{y \in [S] : f(y) \in DFg(x)^{-1}\} \text{ is nonmeager in } [S]) \tag{5.1}$$

since by countability of D this implies that

$$\exists h \in D \exists^* x \in [t]_T \; (\{y \in [S] : f(y)g(x) \in hF\} \text{ is nonmeager in } [S]).$$

But if (5.1) failed, we would have

$$\forall^* x \in [t]_T \; (\{y \in [S] : f(y) \in DFg(x)^{-1}\} \text{ is meager in } [S]).$$

Thus, using the closure of meager sets under countable unions, we would be able to get a $Q \subseteq [t]_T$ dense in $[t]_T$ with $\{y \in [S] : f(y) \in DFg(Q)^{-1}\}$ meager in $[S]$. It would follow that $H \not\subseteq DFg(Q)^{-1}$, contradiction. □

The above characterization from Lemma 5.1 of f-l-extra small sets should be compared with the easy characterization of closed l-small sets as those closed F for which there exists a pruned tree S such that

$$\forall h \in H \forall^* y \in [S] \; f(y) \notin hF.$$

The next lemma registers invariance properties of f-l-extra small sets and points out a difference between this notion and the notion of l-small sets. As seen in Example 2.8 closed l-small sets need not be closed under finite unions. As shown by the lemma below closed f-l-extra small sets always are.

Lemma 5.2. *Let $f : \mathbb{N}^\mathbb{N} \to H$ be a continuous surjection.*

(i) *f-l-extra small sets are invariant under left and right translations by elements of H.*

(ii) *Closed sets that are f-l-extra small form an ideal of closed sets.*

Proof. (i) Invariance from the left is obvious. To prove invariance from the right, let F be f-l-extra small and let $h \in H$. We show that Fh is f-l-extra small. Fix countable $D \subseteq H$ and pruned trees $T_1, \ldots, T_n$. Let $y \in \mathbb{N}^\mathbb{N}$ be such that $f(y) = h^{-1}$. Define T_{n+1} to be the pruned tree $\{y \upharpoonright m : m \in \mathbb{N}\}$ consisting of one branch. Since F is f-l-extra small, there exists $Q \subseteq [T_1] \times \cdots \times [T_n] \times [T_{n+1}]$ somewhere dense and such that

$$H \not\subseteq DF\{f(x_{n+1})^{-1} f(x_n)^{-1} \cdots f(x_1)^{-1} : x \in Q\}. \tag{5.2}$$

Note that in the above formula x_{n+1} is always equal to y, hence $f(x_{n+1})^{-1} = (h^{-1})^{-1} = h$. Thus, it follows from (5.2) that

$$H \not\subseteq D(Fh)\{f(x_n)^{-1} \cdots f(x_1)^{-1} : x \in Q'\},$$

where

$$Q' = \{x' \in [T_1] \times \cdots \times [T_n] : x'^\frown\langle y \rangle \in Q\}.$$

Since Q' is somewhere dense in $[T_1] \times \cdots \times [T_n]$, we have just proved that Fh is f-l-extra small.

(ii) Obviously, if $F_1 \subseteq F_2$ and both of these sets are closed, then if F_2 is f-l-extra small, then so is F_1. Now assume F_1, F_2 are f-l-extra small closed sets. Let a product tree T and a product function g be given. Since F_1 is f-l-extra small, by Lemma 5.1 we can find $t_1 \in T$ and a pruned tree S_1 such that

$$\forall h \in H \forall^* x \in [t_1]_T \ (\{y \in [S_1] : f(y) \in hF_1 g(x)^{-1}\} \text{ is meager in } [S_1]). \tag{5.3}$$

Consider the product tree $S_1 \otimes (T)_{t_1}$ and the product function defined on $[S_1] \times [t_1]_T$ by mapping (y, x) to $f(y)g(x)$. Since F_2 is f-l-extra small, again by Lemma 5.1 there exists a pruned tree S_2, $t_2 \in T$ extending t_1, and $s_1 \in S_1$ such that

$$\begin{aligned} &\forall h \in H \forall^* (y, x) \in [s_1]_{S_1} \times [t_2]_T \\ &\qquad \{z \in [S_2] : f(z) \in hF_2 g(x)^{-1} f(y)^{-1}\} \text{ is meager in } [S_2]. \end{aligned} \tag{5.4}$$

Let $D \subseteq H$ be countable. We will find a countable set $Q \subseteq [T]$ dense in $[t_2]_T$ for which $H \not\subseteq D(F_1 \cup F_2)g(Q)^{-1}$. Fix $P \subseteq [S_2]$ that is countable and dense in $[S_2]$. By (5.3) and the fact that countable intersections of comeager sets are comeager, we get that

$$\begin{aligned}&\forall^* x \in [t_1]_T \forall d \in D \forall p \in P \\ &\qquad \{y \in [S_1] : f(y) \in f(p)^{-1} d F_1 g(x)^{-1}\} \text{ is meager in } [S_1].\end{aligned} \tag{5.5}$$

Using the same argument and (5.4), we get

$$\begin{aligned}&\forall^* (y, x) \in [s_1]_{S_1} \times [t_2]_T \forall d \in D \\ &\qquad \{z \in [S_2] : f(z) \in d F_2 g(x)^{-1} f(y)^{-1}\} \text{ is meager in } [S_2].\end{aligned}$$

We apply the Kuratowski-Ulam theorem to the above statement, to obtain a countable dense in $[s_1]_{S_1}$ set R with the property that for each $r \in R$

$$\begin{aligned}&\forall^* x \in [t_2]_T \forall d \in D \\ &\qquad \{z \in [S_2] : f(z) \in d F_2 g(x)^{-1} f(r)^{-1}\} \text{ is meager in } [S_2].\end{aligned}$$

This implies

$$\begin{aligned}&\forall^* x \in [t_2]_T \forall d \in D \forall r \in R \\ &\qquad \{z \in [S_2] : f(z) \in d F_2 g(x)^{-1} f(r)^{-1}\} \text{ is meager in } [S_2].\end{aligned} \tag{5.6}$$

Noting that $t_1 \subseteq t_2$ and combining (5.5) and (5.6), we see that we can find a countable set Q which is dense in $[t_2]_T$ and is such that for any $q \in Q$, $d \in D$, $p \in P$, and $r \in R$

$$\begin{aligned}&\{y \in [S_1] : f(y) \in f(p)^{-1} d F_1 g(q)^{-1}\} \text{ is meager in } [S_1] \text{ and} \\ &\{z \in [S_2] : f(z) \in d F_2 g(q)^{-1} f(r)^{-1}\} \text{ is meager in } [S_2].\end{aligned} \tag{5.7}$$

We claim that this Q works, that is, that H is not covered by $D(F_1 \cup F_2)g(Q)^{-1}$. Towards a contradiction assume otherwise. Then

$$f([S_2]) f([s_1]_{S_1}) \subseteq D F_1 g(Q)^{-1} \cup D F_2 g(Q)^{-1}. \tag{5.8}$$

Since F_1 and F_2 are closed, this allows us to find $s_2 \in S_2$, $s_1' \in S_1$ with $s_1 \subseteq s_1'$, $d \in D$, and $q \in Q$ such that

$$f([s_2]_{S_2}) f([s_1']_{S_1}) \subseteq d F_1 g(q)^{-1}$$

or

$$f([s_2]_{S_2}) f([s_1']_{S_1}) \subseteq d F_2 g(q)^{-1}.$$

In the first case, pick $p \in P \cap [s_2]_{S_2}$, which is possible since P is dense in $[S_2]$. Then

$$f([s_1']_{S_1}) \subseteq f(p)^{-1} d F_1 g(q)^{-1}.$$

which contradicts the first part of (5.7). In the second case, by density of R in $[s_1]_{S_1}$, we can find an $r \in R \cap [s_1']_{S_1}$. Thus,

$$f([s_2]_{S_2}) \subseteq dF_2 g(q)^{-1} f(r)^{-1}$$

which contradicts the second part of (5.7). □

We recall here the combinatorial description of the equivalence relation $\mathbb{E}_1$ from [22, pp. 160–162]. It will be used in producing an embedding of $\mathbb{E}_1$ into $E_{G/H}$. Fix a sequence (k_n) of natural numbers with the property that for each $k \in \mathbb{N}$, the set $\{n : k_n = k\}$ is infinite and for each n, $k_n \leq 1 + \max\{k_i : i < n\}$. Recursively on n we define equivalence relations E_k on $2^{<\mathbb{N}} = \bigcup_{n\in\mathbb{N}} 2^n$ for all $k \in \mathbb{N}$. This is done as follows. If $s, t \in 2^{<\mathbb{N}}$ and $|s| \neq |t|$, then $\neg(sE_kt)$ for each k. Each E_k is the only possible equivalence relation on the one-element set $2^0 = \{\emptyset\}$. Assuming that all the relations E_k have been defined on $2^0 \cup \cdots \cup 2^{n-1}$, we describe E_k on 2^n as follows:

if $k < k_n$, then for $s, t \in 2^{n-1}$

$$\neg(s^\frown 0 E_k t^\frown 1) \text{ and } (sE_kt \Leftrightarrow s^\frown 0 E_k t^\frown 0 \text{ and } s^\frown 1 E_k t^\frown 1);$$

if $k \geq k_n$, then for $s, t \in 2^{n-1}$

$$sE_kt \Leftrightarrow s^\frown 0 E_k s^\frown 1 E_k t^\frown 0 E_k t^\frown 1.$$

By [22, Lemma 4.1] the equivalence relation on $2^{\mathbb{N}}$ which makes $x_1, x_2 \in 2^{\mathbb{N}}$ equivalent precisely when

$$\exists k \forall n\ (x_1 \restriction n) E_k (x_2 \restriction n) \tag{5.9}$$

is isomorphic to $\mathbb{E}_1$, that is, there exists a homeomorphism between $(2^{\mathbb{N}})^{\mathbb{N}}$ and $2^{\mathbb{N}}$ which transfers $\mathbb{E}_1$ to the above equivalence relation. To make it explicit, we naturally identify $(2^{\mathbb{N}})^{\mathbb{N}}$ with $2^{\mathbb{N}\times\mathbb{N}}$. Then for $y_1, y_2 \in 2^{\mathbb{N}\times\mathbb{N}}$ we have

$$y_1 \mathbb{E}_1 y_2 \text{ iff } \exists k \forall m > k \forall p\ y_1(m,p) = y_2(m,p).$$

Let $\{h(m,p) : p \in \mathbb{N}\}$ be an injective enumeration of $\{n : k_n = m\}$, and define $f : 2^{\mathbb{N}} \to 2^{\mathbb{N}\times\mathbb{N}}$ by letting

$$f(x)(m,p) = x(h(m,p)).$$

Clearly f is a homeomorphism. Furthermore, for $x_1, x_2 \in 2^{\mathbb{N}}$, (5.9) holds precisely when each of the conditions in the following sequence of equivalences holds

$$\begin{aligned} &\exists k \forall n\ (k < k_n \Rightarrow x_1(n) = x_2(n)) \\ &\text{iff } \exists k \forall m > k \forall p\ (x_1(h(m,p)) = x_2(h(m,p))) \\ &\text{iff } \exists k \forall m > k \forall p\ (f(x_1)(m,p) = f(x_2)(m,p)). \end{aligned}$$

The last condition means that $f(x_1) \mathbb{E}_1 f(x_2)$. This allows us to identify in what follows $\mathbb{E}_1$ with the equivalence relation given by (5.9).

It is easy to see [22, Lemma 4.2] that, for each k, $E_k \subseteq E_{k+1}$ and that for each E_k-equivalence class $a \subseteq 2^n$ either

(a) $b = \{s^\frown 0, s^\frown 1 : s \in a\} \subseteq 2^{n+1}$ is an E_k-equivalence class or
(b) $b_0 = \{s^\frown 0 : s \in a\} \subseteq 2^{n+1}$ and $b_1 = \{s^\frown 1 : s \in a\} \subseteq 2^{n+1}$ are two distinct E_k-equivalence classes.

The equivalence classes b, b_0, b_1 as in (a) and (b) will be called *extensions of* a; if (b) holds, b_0 will be sometimes called the 0-*extension of* a and b_1 the 1-*extension of* a.

Proof of Theorem 2.6. Assume that there exists a countable family of closed l-extra small sets covering H. We prove first that there exists a continuous surjection f from $\mathbb{N}^{\mathbb{N}}$ onto H such that a countable family of closed f-l-extra small covers H. Towards a contradiction assume that this is not the case. Let $\mathcal{F}$ be the family of all closed subsets F of G such that for any countable $D, Q \subseteq H$ we have $H \not\subseteq DFQ$. Clearly $\mathcal{F}$ is contained in the family of closed f-l-extra small sets for any f. Now, given f, if, as we assume, countably many closed f-l-extra small sets do not cover H, by Lemma 5.2(i), each closed f-l-extra small is in $\mathcal{F}$. Thus, for each f, the family of closed f-l-extra small sets is equal to $\mathcal{F}$. Therefore, $\mathcal{F}$ is equal to the family of all closed l-extra small sets. Thus, again using our assumption, no countable family of l-closed extra small sets covers H, contradiction.

Fix now f for which there exists a sequence (F_n), $n \in \mathbb{N}$, of closed f-l-extra small sets with $H \subseteq \bigcup_n F_n$. By Lemma 5.2(ii), we can assume that the sequence (F_n) is increasing with respect to inclusion. We will show that $\mathbb{E}_1 \sqsubseteq_c E_{G/H}$. We adopt here the notation concerning $\mathbb{E}_1$ as described in the paragraph preceding this proof. Put

$$M_n = \max\{k_i : i \leq n\}.$$

At stage n we will have, for each $i \leq M_n$, a pruned product tree T_i, a product function $g_i : [T_i] \to H$, and $\tau_a \in T_i$ where a is an E_i-equivalence class. We will also have open sets $U_t \subseteq G$, $t \in 2^{\leq n}$. These objects will fulfill the following conditions, where d is a complete metric on G.

(i) For $n \geq 1$, if $s \in 2^{n-1}$, $t \in 2^n$ and $s \subseteq t$, then $\overline{U_t} \subseteq U_s$.

(ii) For $s \in 2^n$, the d-diameter of U_s is $\leq 1/(n+1)$.

(iii) For $n \geq 1$, if b is an E_i-equivalence class on 2^n with $i \leq M_{n-1}$ and it extends an E_i-equivalence class $a \subseteq 2^{n-1}$, then $\tau_a \subsetneq \tau_b$.

(iv) For $n \geq 1$, for $s_1, s_2 \in 2^n$ if $s_1 \restriction (n-1) E_{k_{n-1}} s_2 \restriction (n-1)$ and $s_1(n-1) = 1$, $s_2(n-1) = 0$, then $U_{s_1}^{-1} U_{s_2} \cap F_{k_n - 1} = \emptyset$.

(v) Let $s \in 2^n$ and let $b_M \supseteq b_{M-1} \supseteq \cdots \supseteq b_0$, $M = M_n$, be the equivalence classes in 2^n containing s where b_i is an E_i-equivalence class. Then

$$\forall (x_M, \ldots, x_0) \in [\tau_{b_M}]_{T_M} \times \cdots \times [\tau_{b_0}]_{T_0} \; g_M(x_M) \cdots g_0(x_0) \in U_s.$$

(vi) Let $1 \leq k \leq M_n$. For any $b_{k-1} \supseteq b_{k-2} \supseteq \cdots \supseteq b_0$, with b_j an E_j-equivalence class on 2^n, $0 \leq j < k$, we have

$$\forall h \in H \forall^* (x_{k-1}, \ldots, x_0) \in [\tau_{b_{k-1}}]_{T_{k-1}} \times \cdots \times [\tau_{b_0}]_{T_0}$$
$$\{x \in [T_k] : g_k(x) \in h F_k g_0(x_0)^{-1} \cdots g_{k-1}(x_{k-1})^{-1}\} \text{ is meager in } [T_k].$$

Assuming that the construction has been carried out, define $\phi : 2^{\mathbb{N}} \to G$ by letting $\phi(x)$ be the unique element in $\bigcap_n U_{x \restriction n}$. From (i) and (ii), we get that ϕ is well-defined and continuous. We will now find a different representation for $\phi(x)$. For $k \leq M_n$, let b_k^n be the E_k-equivalence class containing $x \restriction n$. Then b_k^{n+1} is an extension of b_k^n and, therefore, $\tau_{b_k^n} \subsetneq \tau_{b_k^{n+1}}$ by (iii). So we can find a unique $\alpha_k^x \in [T_k]$ such that $\tau_{b_k^n} \subseteq \alpha_k^x$ for all n with $k \leq M_n$. We claim that

$$\phi(x) = \lim_k g_k(\alpha_k^x) g_{k-1}(\alpha_{k-1}^x) \cdots g_0(\alpha_0^x). \tag{5.10}$$

Note that the increasing sequence (M_n) attains each positive integer value. Therefore, (5.10) follows from the fact that for each n we have

$$g_{M_n}(\alpha_{M_n}^x) g_{M_n - 1}(\alpha_{M_n - 1}^x) \cdots g_0(\alpha_0^x) \in U_{x \restriction n}.$$

This, in turn, is immediate from (v) applied to $s = x \restriction n$ and $\alpha_M^x \in [\tau_{b_M^n}]_{T_M}$, $\alpha_{M-1}^x \in [\tau_{b_{M-1}^n}]_{T_{M-1}}$, ... , $\alpha_0^x \in [\tau_{b_0^n}]_{T_0}$ where $M = M_n$.

If $x \mathbb{E}_1 y$, then for some k and for all n, $x \restriction n E_k y \restriction n$. Thus, for each n, $x \restriction n$ and $y \restriction n$ belong to the same E_k-equivalence class, so for all $l \geq k$,

$\alpha_l^x = \alpha_l^y$. It follows from (5.10) that

$$\phi(x)^{-1}\phi(y) = g_0(\alpha_0^x)^{-1} \cdots g_{k-1}(\alpha_{k-1}^x)^{-1} g_{k-1}(\alpha_{k-1}^y) \cdots g_0(\alpha_0^y) \in H.$$

Now assume that $\neg(x\mathbb{E}_1 y)$. This allows us to find an infinite sequence $(n_l)_{l\in\mathbb{N}}$, so that $k_{n_l} < k_{n_{l+1}}$, $x \restriction (n_l - 1) E_{k_{n_l}-1} y \restriction (n_l - 1)$, and $x(n_l - 1) \neq y(n_l - 1)$. By going to a subsequence, we can assume that $x(n_l - 1) = 1$ and $y(n_l - 1) = 0$ for all l or that $x(n_l - 1) = 0$ and $y(n_l - 1) = 1$ for all l. Assume without loss of generality that the first alternative holds. By definition of ϕ, $\phi(x) \in U_{x\restriction n_l}$ and $\phi(y) \in U_{y\restriction n_l}$. By (iv), we see that $\phi(x)^{-1}\phi(y) \notin F_{k_{n_l}-1}$. Since the sequence (F_k) is increasing, $\phi(x)^{-1}\phi(y) \notin \bigcup_k F_k$, whence $\neg(\phi(x) E_{G/H} \phi(y))$.

It remains to carry out the construction. For $n = 0$, $M_0 = k_0 = 0$. Let $g_0 = f$, $T_0 = \{x \restriction n : n \in \mathbb{N}\}$ for some $x \in \mathbb{N}^{\mathbb{N}}$, $\tau_{\{\emptyset\}} = \emptyset$ ($\{\emptyset\}$ is the only E_0-equivalence class on 2^0), and let $U_\emptyset$ be some open subset of G with d-diameter ≤ 1 containing $f(x)$. Conditions (i), (iii), (iv), and (vi) are void for $n = 0$ while (ii) and (v) are trivially true. Assume therefore that everything has been defined up to and including n.

We now carry out step $n+1$. We will be using the formulation of f-l-extra smallness from Lemma 5.1.

Claim 1. Put $M = M_n$. Let $k \leq M$. Let finitely many elements $h_0, \ldots, h_p$ of H be given. For each E_i-equivalence class b on 2^n, $i \leq k-1$, there exists $x_b \in [\tau_b]_{T_i}$ such that for any sequence $b_{k-1} \supseteq \cdots \supseteq b_0$, where b_i is an E_i-equivalence class and any h_j, $j \leq p$, we have

$$\forall^* x \in [T_k]\ g_k(x) \notin h_j F_k g_0(x_{b_0})^{-1} \cdots g_{k-1}(x_{b_{k-1}})^{-1}.$$

Proof of Claim 1. Fix $h_0, \ldots, h_p \in H$. Below b_i or $\bar{b}_i$ stands for an E_i-equivalence class on 2^n. Fix $\bar{b}_{k-1}$. Then, by (vi), we have

$$\forall b_0 \subseteq \cdots \subseteq b_{k-2} \subseteq \bar{b}_{k-1} \forall^*(x_{k-1}, \ldots, x_0) \in [\tau_{\bar{b}_{k-1}}]_{T_{k-1}} \times \cdots \times [\tau_{b_0}]_{T_0}$$
$$\forall j \leq p \forall^* x \in [T_k]\ g_k(x) \notin h_j F_k g_0(x_0)^{-1} \cdots g_{k-1}(x_{k-1})^{-1}.$$

It follows that for any sequence $b_0 \subseteq \cdots \subseteq b_{k-2} \subseteq \bar{b}_{k-1}$, we can apply the Kuratowski-Ulam theorem to infer that

$$\forall^* x_{k-1} \in [\tau_{\bar{b}_{k-1}}]_{T_{k-1}} \forall^*(x_{k-2}, \ldots, x_0) \in [\tau_{b_{k-2}}]_{T_{k-1}} \times \cdots \times [\tau_{b_0}]_{T_0}$$
$$\forall j \leq p \forall^* x \in [T_k]\ g_k(x) \notin h_j F_k g_0(x_0)^{-1} \cdots g_{k-1}(x_{k-1})^{-1}.$$

This implies that for a comeager set of $x_{k-1} \in [\tau_{\bar{b}_{k-1}}]_{T_{k-1}}$ we have

$$\begin{aligned} &\forall b_0 \subseteq \cdots \subseteq b_{k-2} \subseteq \bar{b}_{k-1} \\ &\forall^*(x_{k-2},\dots,x_0) \in [\tau_{b_{k-2}}]_{T_{k-2}} \times \cdots \times [\tau_{b_0}]_{T_0} \forall j \leq p \forall^* x \in [T_k] \\ &\qquad g_k(x) \notin h_j F_k g_0(x_0)^{-1} \cdots g_{k-2}(x_{k-2})^{-1} g_{k-1}(x_{k-1})^{-1}. \end{aligned} \tag{5.11}$$

Pick $x_{\bar{b}_{k-1}}$ from this comeager set of x_{k-1}'s. Thus, we found x_b for each E_{k-1}-equivalence class on 2^n. Fix now $\bar{b}_{k-2}$. Let $\bar{b}_{k-1}$ be the unique E_{k-1}-equivalence class containing $\bar{b}_{k-2}$. Use now

$$\begin{aligned} &\forall b_0 \subseteq \cdots \subseteq b_{k-3} \subseteq \bar{b}_{k-2} \forall^*(x_{k-2},\dots,x_0) \in [\tau_{\bar{b}_{k-2}}]_{T_{k-2}} \times \cdots \times [\tau_{b_0}]_{T_0} \\ &\qquad \forall j \leq p \forall^* x \in [T_k] \\ &\qquad\qquad g_k(x) \notin h_j F_k g_0(x_0)^{-1} \cdots g_{k-2}(x_{k-2})^{-1} g_{k-1}(x_{\bar{b}_{k-1}})^{-1}, \end{aligned}$$

which follows from (5.11), to repeat the above argument and find $x_{\bar{b}_{k-2}}$ so that

$$\begin{aligned} &\forall b_0 \subseteq \cdots \subseteq b_{k-3} \subseteq \bar{b}_{k-2} \\ &\quad \forall^*(x_{k-3},\dots,x_0) \in [\tau_{b_{k-3}}]_{T_{k-3}} \times \cdots \times [\tau_{b_0}]_{T_0} \forall j \leq p \forall^* x \in [T_k] \\ &\quad g_k(x) \notin h_j F_k g_0(x_0)^{-1} \cdots g_{k-3}(x_{k-3})^{-1} g_{k-2}(x_{\bar{b}_{k-2}})^{-1} g_{k-1}(x_{\bar{b}_{k-1}})^{-1}. \end{aligned}$$

In fact, we get an appropriate x_b for each E_{k-2}-equivalence class b on 2^n. We continue in this fashion to obtain points x_b for all the E_i-equivalence classes with $i \leq k-1$ so that the condition from the conclusion of the claim is fulfilled. This finishes the proof of the claim.

Claim 2. Put $M = M_n$ and $k = k_{n+1} - 1$. (Note that $k \leq M$.) For $s \in 2^{n+1}$ there exists $U_s \subseteq G$ open and for any b an E_i-equivalence class on 2^{n+1}, $i \leq M$, there exists $\tau'_b \in T_i$ so that

(i') For $s \in 2^{n+1}$, $\overline{U_s} \subseteq U_{s\restriction n}$.

(ii') For $s \in 2^{n+1}$, the d-diameter of U_s is $\leq 1/(n+2)$.

(iii') If b is an E_i-equivalence class on 2^{n+1}, $i \leq M$, and it extends an E_i-equivalence class $a \subseteq 2^n$, then $\tau_a \subsetneq \tau'_b$.

(iv') For $s_1, s_2 \in 2^{n+1}$ if $s_1 \restriction n E_k s_2 \restriction n$ and $s_1(n) = 1$, $s_2(n) = 0$, then $U_{s_1}^{-1} U_{s_2} \cap F_k = \emptyset$.

(v') Let $s \in 2^{n+1}$ and let $b_M \supseteq b_{M-1} \supseteq \cdots \supseteq b_0$ be equivalence classes in 2^{n+1} containing s with b_i being the E_i-equivalence class. Then

$$\forall (x_M,\dots,x_0) \in [\tau'_{b_M}]_{T_M} \times \cdots \times [\tau'_{b_0}]_{T_0} \; g_M(x_M) \cdots g_0(x_0) \in U_s.$$

Proof of Claim 2. We will first define U_s for $s \in 2^{n+1}$. Pick $\bar{x}_b \in [\tau_b]_{T_i}$ for each E_i-equivalence class b on 2^n, $i \leq M$. Let a be an E_k-equivalence class on 2^n with k as in the assumption of the claim. Below b_i, b'_i stand for E_i-equivalence classes on 2^n. Consider the finite set of elements of H of the following form $g_k(\bar{x}_a)g_{k-1}(\bar{x}_{b'_{k-1}})\cdots g_0(\bar{x}_{b'_0})$ for all possible sequences $b'_0 \subseteq \cdots \subseteq b'_{k-1} \subseteq a$. An application of Claim 1 to this set gives us points $x_b \in [\tau_b]_{T_i}$, for each E_i-equivalence class b on 2^n with $i < k$. These points have the property that the set

$$R = \bigcup_{b_{k-1}\supseteq\cdots\supseteq b_0} \bigcup_{b'_{k-1}\supseteq\cdots\supseteq b'_0} \{x \in [T_k] : g_k(x) \in g_k(\bar{x}_a)g_{k-1}(\bar{x}_{b'_{k-1}})\cdots g_0(\bar{x}_{b'_0})F_k g_0(x_{b_0})^{-1}\cdots g_{k-1}(x_{b_{k-1}})^{-1}\}$$

is meager in $[T_k]$. Pick $\bar{y}_a \in [\tau_a]_{T_k} \setminus R$.

By (v) for n, for any $t \in 2^n$ and for $b_M \supseteq \cdots \supseteq b_0 = \{t\}$, we have

$$g_M(\bar{x}_{b_M})\cdots g_0(\bar{x}_{b_0}) \in U_t \text{ and } g_M(\bar{x}_{b_M})\cdots g_{k+1}(\bar{x}_{b_{k+1}})g_k(\bar{y}_{b_k})g_{k-1}(x_{b_{k-1}})\cdots g_0(x_{b_0}) \in U_t. \tag{5.12}$$

Note that if $t, s \in 2^n$ are E_k related, they belong to the same E_i-equivalence classes for $k \leq i \leq M$. Thus, from the choice of $\bar{y}_a$, we get that for such s and t and for $b_M \supseteq \cdots \supseteq b_0 = \{t\}$ and $b'_M \supseteq \cdots \supseteq b'_0 = \{s\}$, we have

$$\big(g_M(\bar{x}_{b_M})\cdots g_{k+1}(\bar{x}_{b_{k+1}})g_k(\bar{y}_{b_k})g_{k-1}(x_{b_{k-1}})\cdots g_0(x_{b_0})\big)^{-1}g_M(\bar{x}_{b'_M})\cdots g_0(\bar{x}_{b'_0}) = g_0(x_{b_0})^{-1}\cdots g_{k-1}(x_{b_{k-1}})^{-1}g_k(\bar{y}_{b_k})^{-1}g_k(\bar{x}_{b'_k})\cdots g_k(\bar{x}_{b'_0}) \notin F_k.$$

Since this condition is open, using (5.12), we can find for any $t \in 2^n$, $U_{t^\frown 0}$, $U_{t^\frown 1}$ such that

$$\overline{U}_{t^\frown 0}, \overline{U}_{t^\frown 1} \subseteq U_t \tag{5.13}$$

$$\operatorname{diam}(U_{t^\frown 0}), \operatorname{diam}(U_{t^\frown 1}) < 1/(n+2) \tag{5.14}$$

$$\forall b_M \supseteq \cdots \supseteq b_0 = \{t\}\ g_M(\bar{x}_{b_M})\cdots g_0(\bar{x}_{b_0}) \in U_{t^\frown 1} \text{ and } g_M(\bar{x}_{b_M})\cdots g_k(\bar{y}_{b_k})g_{k-1}(x_{b_{k-1}})\cdots g_0(x_{b_0}) \in U_{t^\frown 0} \tag{5.15}$$

$$U_{s^\frown 1}^{-1}U_{t^\frown 0} \cap F_k = \emptyset \text{ if } sE_kt. \tag{5.16}$$

Note that (5.13), (5.14), and (5.16) guarantee that (i'), (ii'), and (iv') are fulfilled. Let c be an E_i-equivalence class on 2^{n+1}. If $i \leq k$ and c is the 1-extension of b, an E_i-equivalence class on 2^n, we will let τ'_c be an initial segment of $\bar{x}_b$. If $i \leq k$ and c is the 0-extension of b, then τ'_c will be an initial segment of $\bar{y}_b$ if $i = k$, and an initial segment of x_b if $i < k$. If $i > k$ and c is the extension of b, then τ'_c will be an initial segment of $\bar{x}_b$. The only condition on these initial segments is that they are long enough so that (iii') holds and, using (5.15), so that (v') holds as well. The claim is proved.

Case 1. $k_{n+1} \leq M_n$

In this case $M_{n+1} = M_n$. Let M stand for this common value. Apply Claim 2 with $k = k_{n+1} - 1$, and let for b an E_i-equivalence class on 2^{n+1}, $i \leq M$, $\tau_b = \tau'_b$. Note that (i)–(v) for $n+1$ are implied by Claim 2(i')–(v'), respectively. Point (vi) for $n+1$ holds by (vi) for n and by (iii').

Case 2. $k_{n+1} = M_n + 1$

Note that $M_{n+1} = k_{n+1}$. Again put $M = M_n$. Apply Claim 2 with $k = k_{n+1} - 1 = M$ to obtain τ'_b for E_i-equivalence classes b on 2^{n+1} with $i \leq M$. We will define τ_b for such b as appropriate extensions of τ'_b. We still need to define T_{M+1} and τ_a for the unique E_{M+1}-equivalence class a on 2^{n+1}. Now we proceed to the construction of these objects.

Enumerate by $i \leq I$, for some $I \in \mathbb{N}$, all possible sequences

$$b^i_M \supseteq b^i_{M-1} \supseteq \cdots \supseteq b^i_0$$

where b^i_l is an E_l-equivalence class on 2^{n+1}. We perform now a construction by induction on $i \leq I$. At stage i, we will have produced for all $j \leq i$, $\tau^j_b \in T_l$, for each b an E_l-equivalence class with $l \leq M$, S_j a pruned tree on $\mathbb{N}$, $\sigma^i_j \in S_j$, and $q_j \in H$. Put $f_j(x) = q_j \cdot f(x)$. These objects will have the following properties. (Below $j, j_1, j_2 \leq i$, b is an E_l-equivalence class on 2^{n+1} for some $l \leq M$; if we want to make the $l \leq M$ explicit, we write b_l for b.)

(a) $\sigma^{i-1}_j \subseteq \sigma^i_j$, for $i > 0$.
(b) $\tau'_b \subseteq \tau^{i-1}_b \subseteq \tau^i_b$, for $i > 0$.
(c) For any $h \in H$ there exists $A_h \subseteq [\sigma^i_{i-1}]_{S_{i-1}} \times \cdots \times [\sigma^i_0]_{S_0} \times [\tau^i_{b^i_M}]_{T_M} \times \cdots \times [\tau^i_{b^i_0}]_{T_0}$ comeager and such that

$$\forall (y_{i-1}, \ldots, y_0, x_M, \ldots, x_0) \in A_h \forall^* w \in [S_i]$$
$$f(w) \notin hF_{M+1} g_0(x_0)^{-1} \cdots g_M(x_M)^{-1} f_0(y_0)^{-1} \cdots f_{i-1}(y_{i-1})^{-1}.$$

(d) For any sequence $b_M \supseteq \cdots \supseteq b_0 = \{t\}$ with $t \in 2^{n+1}$, we have

$$\forall y \in [\sigma^i_i]_{S_i} \times [\sigma^i_{i-1}]_{S_{i-1}} \times \cdots \times [\sigma^i_0]_{S_0} \forall x \in [\tau^i_{b_M}]_{T_M} \times \cdots \times [\tau^i_{b_0}]_{T_0}$$
$$f_i(y_i) \cdots f_0(y_0) g_M(x_M) \cdots g_0(x_0) \in U_t,$$

where $y = (y_i, \ldots, y_0)$ and $x = (x_M, \ldots, x_0)$.
(e) $|\sigma^i_{j_1}| = |\sigma^i_{j_2}|$.

If $i = 0$, by convention, in (c) above, $A_h \subseteq [\tau^0_{b^0_M}]_{T_M} \times \cdots \times [\tau^0_{b^0_0}]_{T_0}$.

Assume the construction has been carried out up to stage $i \geq 0$ or nothing has happened yet and then put $i = -1$ and $\tau_b^{-1} = \tau'_b$ for any E_l-equivalence class b on 2^{n+1} with $l \leq M$. Consider the product tree $(S_i)_{\sigma^i_i} \otimes \cdots \otimes (S_0)_{\sigma^i_0} \otimes (T_M)_{\tau^i_{b^{i+1}_M}} \otimes \cdots \otimes (T_0)_{\tau^i_{b^{i+1}_0}}$ (if $i = -1$, the tree is defined to be $(T_M)_{\tau^{-1}_{b^0_M}} \otimes \cdots \otimes (T_0)_{\tau^{-1}_{b^0_0}}$) and the product function defined on the set of all branches through that tree by

$$(y_i, \ldots, y_0, x_M, \ldots, x_0) \to f_i(y_i) \cdots f_0(y_0) g_M(x_M) \cdots g_0(x_0).$$

Since F_{M+1} is f-l-extra small, by Lemma 5.1, we can find a pruned tree S_{i+1} on $\mathbb{N}$ and $\bar{\sigma}_i, \ldots, \bar{\sigma}_0$ and $\bar{\tau}_{b^{i+1}_M}, \ldots, \bar{\tau}_{b^{i+1}_0}$ extensions of $\sigma^i_i, \ldots, \sigma^i_0$ and $\tau^i_{b^{i+1}_M}, \ldots, \tau^i_{b^{i+1}_0}$ in $S_i, \ldots, S_0$ and $T_M, \ldots, T_0$, respectively. These objects have the property that for each $h \in H$ there exists a set

$$A'_h \subseteq [\bar{\sigma}_i]_{S_i} \times \cdots \times [\bar{\sigma}_0]_{S_0} \times [\bar{\tau}_{b^{i+1}_M}]_{T_M} \times \cdots \times [\bar{\tau}_{b^{i+1}_0}]_{T_0}$$

which is comeager and such that for any element $(y_i, \ldots, y_0, x_M, \ldots, x_0)$ of it

$$\begin{aligned} &\forall^* w \in [S_{i+1}] \\ &\qquad f(w) \notin h F_{M+1} g_0(x_0)^{-1} \cdots g_M(x_M)^{-1} f_0(y_0)^{-1} \cdots f_i(y_i)^{-1}. \end{aligned} \tag{5.17}$$

For all the τ^i_b's not involved above, let $\bar{\tau}_b = \tau^i_b$. Let $\bar{w}$ be in S_{i+1}, and let $q_{i+1} = f(\bar{w})^{-1} \in H$ so that we have

$$\begin{aligned} &\forall b_M \supseteq \cdots \supseteq b_0 = \{t\} \\ &\quad \forall y \in [\bar{\sigma}_i]_{S_i} \times \cdots \times [\bar{\sigma}_0]_{S_0} \forall x \in [\bar{\tau}_{b_M}]_{T_M} \times \cdots \times [\bar{\tau}_{b_0}]_{T_0} \\ &\qquad q_{i+1} f(\bar{w}) f_i(y_i) \cdots f_0(y_0) g_M(x_M) \cdots g_0(x_0) \in U_t, \end{aligned} \tag{5.18}$$

where $y = (y_i, \ldots, y_0)$ and $x = (x_M, \ldots, x_0)$. This is possible by (d) if $i \geq 0$ and by Claim 2(v') if $i = -1$. Define f_{i+1} to be $q_{i+1} f$. Now use (5.18) to get σ^{i+1}_j for $j < i+1$ and τ^{i+1}_b for b an E_l-equivalence class for $l \leq M$ by extending $\bar{\sigma}_j$ and $\bar{\tau}_b$ and to find σ^{i+1}_{i+1}, a long enough initial segment of $\bar{y}$, so that (d) holds. Obviously, we can make sure that all the σ sequences have equal length taking care of (e). For each $h \in H$, let

$$A_h = A'_h \cap [\sigma^{i+1}_i]_{S_i} \times \cdots \times [\sigma^{i+1}_0]_{S_0} \times [\tau^{i+1}_{b^{i+1}_M}]_{T_M} \times \cdots \times [\tau^{i+1}_{b^{i+1}_0}]_{T_0}.$$

Note that (c) is fulfilled by (5.17). This finishes our inductive construction.

Finally, define T_{M+1} to be the product tree on $\mathbb{N}^I$ given by

$$(S_I)_{\sigma^I_I} \otimes (S_{I-1})_{\sigma^I_{I-1}} \otimes \cdots \otimes (S_0)_{\sigma^I_0}.$$

Define also a product function g_{M+1} on $[T_{M+1}]$ by letting

$$g_{M+1}(y_I, \ldots, y_0) = f_I(y_I) \cdots f_0(y_0).$$

Further let $\tau_b = \tau_b^I$ for any b an E_l-equivalence class on 2^{n+1} for $l \leq M$. Let τ_a be the node in T_{M+1} which is identified via the standard coding with $\langle \sigma_I^I, \ldots, \sigma_0^I \rangle$ for the only E_{M+1}-equivalence class a on 2^{n+1}. This is a node in T_{M+1} by (e).

Properties (i)–(iv) follow from Claim 2(i')–(iv'). Property (v) is an immediate consequence of (d) for $i = I$. Thus, it suffices to verify (vi). Moreover, by (iii) for $n+1$ and (vi) for n, (vi) for $n+1$ holds for any $k \leq M_n = M$. Therefore, we only need to check it for $k = M+1$. So let $b_M \supseteq \cdots \supseteq b_0$ be a sequence of equivalence classes on 2^{n+1} with b_j being an E_j-equivalence class. Let h_0 be an element of H. We need to show that

$$\begin{aligned} &\forall^* x \in [\tau_{b_M}]_{T_M} \times \cdots \times [\tau_{b_0}]_{T_0} \forall^* y \in [T_{M+1}] \\ &\qquad\qquad g_{M+1}(y) \notin h_0 F_{M+1} g_0(x_0)^{-1} \cdots g_M(x_M)^{-1}, \end{aligned} \tag{5.19}$$

where $x = (x_M, \ldots, x_0)$. Assume that the sequence $b_M \supseteq \cdots \supseteq b_0$ appeared in our listing of such sequences at place $i \leq I$. By point (c) applied to

$$h = q_i^{-1} f_{i+1}(z_{i+1})^{-1} \cdots f_I(z_I)^{-1} h_0,$$

we get that for any $z = (z_I, \ldots, z_{i+1})$ from $[\sigma_I^I]_{S_I} \times \cdots \times [\sigma_{i+1}^I]_{S_{i+1}}$,

$$\begin{aligned} &\forall^*(y,x) \in ([\sigma_{i-1}^i]_{S_{i-1}} \times \cdots \times [\sigma_0^i]_{S_0}) \times ([\tau_{b_M}^i]_{T_M} \times \cdots \times [\tau_{b_0}^i]_{T_0}) \\ &\qquad \forall^* w \in [S_i]\ f(w) \notin q_i^{-1} f_{i+1}(z_{i+1})^{-1} \cdots f_I(z_I)^{-1} h_0 F_{M+1} \times \\ &\qquad\qquad g_0(x_0)^{-1} \cdots g_M(x_M)^{-1} f_0(y_0)^{-1} \cdots f_{i-1}(y_{i-1})^{-1}, \end{aligned} \tag{5.20}$$

where $(y,x) = (y_{i-1}, \ldots, y_0, x_M, \ldots, x_0)$. Note that the condition

$$\begin{aligned} &f(w) \notin q_i^{-1} f_{i+1}(z_{i+1})^{-1} \cdots f_I(z_I)^{-1} h_0 F_{M+1} \times \\ &\qquad g_0(x_0)^{-1} \cdots g_M(x_M)^{-1} f_0(y_0)^{-1} \cdots f_{i-1}(y_{i-1})^{-1} \end{aligned}$$

is open on (z,y,x,w) so has the Baire property and we can apply the Kuratowski-Ulam theorem to (5.20). After doing so we obtain

$$\begin{aligned} &\forall^* x \in [\tau_{b_M}^i]_{T_M} \times \cdots \times [\tau_{b_0}^i]_{T_0} \forall^* y \in [\sigma_{i-1}^i]_{S_{i-1}} \times \cdots \times [\sigma_0^i]_{S_0} \\ &\quad \forall^* w \in [S_i] \forall^* z \in [\sigma_I^I]_{S_I} \times \cdots \times [\sigma_{i+1}^I]_{S_{i+1}} \\ &\qquad f(w) \notin q_i^{-1} f_{i+1}(z_{i+1})^{-1} \cdots f_I(z_I)^{-1} h_0 F_{M+1} \times \\ &\qquad\quad g_0(x_0)^{-1} \cdots g_M(x_M)^{-1} f_0(y_0)^{-1} \cdots f_{i-1}(y_{i-1})^{-1}. \end{aligned}$$

But this implies by (a) and (b) that on a comeager set of $x \in [\tau_{b_M}]_{T_M} \times \cdots \times [\tau_{b_0}]_{T_0}$ we have

$$\begin{aligned}\forall^*(z, w, y) \in [\sigma_I^I]_{S_I} \times \cdots \times [\sigma_i^I]_{S_i} \times \cdots \times [\sigma_0^I]_{S_0} \\ g_{M+1}(z_I, \dots, z_{i+1}, w, y_{i-1}, \dots, y_0) = f_I(z_I) \cdots q_i f(w) \cdots f_0(y_0) \\ \notin h_0 F_{M+1} g_0(x_0)^{-1} \cdots g_M(x_M)^{-1},\end{aligned}$$

where $(z, w, y) = (z_I, \dots, z_{i+1}, w, y_{i-1}, \dots, y_0)$. Thus, (5.19) is proved and so also is (vi). □

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