AVOIDING FAMILIES AND TUKEY FUNCTIONS ON THE NOWHERE DENSE IDEAL

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Abstract. We investigate Tukey functions from the ideal of all closed nowhere dense subsets of $2^{\mathbb{N}}$. In particular, we answer an old question of Isbell and Fremlin by showing that this ideal is not Tukey reducible to the ideal of density zero subsets of $\mathbb{N}$. We also prove non-existence of various special types of Tukey reductions from the nowhere dense ideal to analytic P-ideals. In connection with these results, we study families $\mathcal{F}$ of clopen subsets of $2^{\mathbb{N}}$ with the property that for each nowhere dense subset of $2^{\mathbb{N}}$ there is a set in $\mathcal{F}$ not intersecting it. We call such families avoiding.

1. Introduction

Let D, E with binary relations $\leq_D$ and $\leq_E$, respectively, be directed partial orders. A function $f\colon D \to E$ is called *Tukey* if for each $e \in E$ there is $d \in D$ such that $f(x) \leq_E e$ implies $x \leq_D d$, that is, preimages of bounded sets are bounded. If such a Tukey function from D to E exists, we write $D \leq_T E$ and say that D is *Tukey reducible to* E. If both $D \leq_T E$ and $E \leq_T D$, we write $D \equiv_T E$ and say that D and E are *Tukey equivalent*. Two directed orders are Tukey equivalent precisely when they are isomorphic to cofinal subsets of the same directed order. For more background on Tukey reductions, we direct the reader to [3] and [14] and for applications of this notion in the study of partial orders, in addition to these two papers, the reader may consult [4], [5], [7], [8], [9], [13], [15], and [17].

Before outlining our results, we recall the background and history of one of the questions on Tukey reductions that we consider in this paper, which will give us an opportunity to recall notions and notation standard in the area. Isbell [5] initiated the study of Tukey reductions among directed orders coming from analysis and topology. He isolated four such orders:

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— $\mathbb{N}^{\mathbb{N}}$, the set of all functions from $\mathbb{N}$ to $\mathbb{N}$ with the order equal to the pointwise inequality between such functions;
— NWD, the ideal of closed nowhere dense subsets of $2^{\mathbb{N}}$ with inclusion as the order relation;
— $\mathcal{Z}_0$, the ideal of subsets of $\mathbb{N}$ of density zero with inclusion, where $x \subseteq \mathbb{N}$ has density zero if $\lim_n |x \cap \{0, \ldots, n\}|/(n+1) = 0$;
— ℓ_1, the ideal of those subsets x of $\mathbb{N}$ for which $\sum_{n \in x} 1/(n+1) < \infty$ again taken with the inclusion relation.

Isbell proved in [5] that $\mathbb{N}^{\mathbb{N}}$ is Tukey below the other three orders, with $\mathcal{Z}_0$ and ℓ_1 strictly Tukey above it, and asked for the complete determination of Tukey reductions among the orders above. Later, Fremlin [3] added to the picture the directed order

— $\mathcal{E}_\mu$, the ideal of all compact Lebesgue measure zero subsets of $2^{\mathbb{N}}$ with inclusion as the order relation.

He proved in [3] the following inequalities

$$\mathbb{N}^{\mathbb{N}} \leq_T \mathcal{E}_\mu,$$
$$\mathcal{E}_\mu \leq_T \mathrm{NWD} \leq_T \ell_1,$$
$$\mathcal{E}_\mu \leq_T \mathcal{Z}_0 \leq_T \ell_1.$$

The proof of NWD $\leq_T$ ℓ_1 drew on an earlier important insight due to Bartoszyński [1] and Raissonier–Stern [11] and some technical ideas of Pawlikowski. Fremlin [3, 3M] raised the problem of determining weather the Tukey reductions in the picture above are all the Tukey reductions among the five orders involved in it. A moment's thought convinces one that to prove that this is so, it suffices to show that

$$\mathbb{N}^{\mathbb{N}} \not\geq_T \mathcal{E}_\mu, \tag{1a}$$
$$\mathrm{NWD} \not\geq_T \mathcal{Z}_0, \tag{1b}$$
$$\mathcal{Z}_0 \not\geq_T \mathrm{NWD}. \tag{1c}$$

Inequalities (1a) and (1b) were again proved by Fremlin in [3, 3M], while (1c) remained open [3, 3N(a)]. More recently, two non-reduction results that follow from (1c) were proved by Louveau and Veličković [7], who showed

$$\mathcal{Z}_0 \not\geq_T \ell_1, \tag{2}$$

and by Mátrai, see [8], who showed

$$\mathcal{E}_\mu \not\geq_T \mathrm{NWD}.$$

We complete this line of research with Theorem 3.5, which establishes (1c). (We were informed by Tamás Mátrai that he also independently proved (1c). His proof will be published in a separate paper.)

Our study of Tukey functions defined on NWD is contained in Section 3. In [14], inequalities (1a), (1b) and (2) were extended and placed in a wider context. Similarly here in Theorem 3.5, Subsection 3.3, we prove not only (1c) but a broader non-reduction result: we show that NWD does not Tukey reduce to any density-like analytic P-ideal. (For the definition of density-like ideals see Subsection 3.1; the ideal $\mathcal{Z}_0$ is in this class.) The proof of this theorem combines combinatorial considerations concerning avoiding families (see the next paragraph and Section 2) with topological arguments. These arguments use results from [14] and the notion of the Ochan topology from [16] and [10] that is developed further in Subsection 3.2. On the other hand, also in Subsection 3.3, adapting methods of Fremlin from [3], we show (Theorem 3.7(i)) that NWD does have a Tukey reduction, even a continuous one, to all summable-like ideals. (For definition of summable-like ideals see Subsection 3.1; the ideal ℓ_1 is in this class.) We also show that all summable-like ideals are Tukey equivalent to each other (Corollary 3.8). Furthermore, in Subsection 3.4, we rule out some special types of Tukey reductions from NWD to analytic P-ideals: those mapping bounded sets to bounded sets (Theorem 3.9) and those produced with the use of a complete metric making the union operation uniformly continuous (Theorem 3.10).

One of our main tools in proving Theorem 3.5 mentioned above is the notion of an avoiding family. Our study of such families is described in Section 2. By an avoiding family we understand a collection of clopen subsets of $2^{\mathbb{N}}$ such that each nowhere dense set is disjoint from some clopen set from the collection. We investigate functions from avoiding families $\mathcal{F}$ (or sequences of such families) to sets Z. There is a natural notion of a large subset of an avoiding family $\mathcal{F}$: these are those subsets of $\mathcal{F}$ that have empty intersection. By avoidance the whole family has this property and by compactness each subset with empty intersection contains a finite such subset. Any set Z can be equipped with a family $\mathcal{B}$ of its finite subsets such that $\mathcal{B}$ is closed under taking subsets and it contains all singletons of Z. We call such families $\mathcal{B}$ collectives and consider sets contained in $\mathcal{B}$ as small subsets of Z. We make assumptions ensuring that $\mathcal{B}$ is rich enough, for example, for each infinite subset X of Z there is an infinite subset Y of X whose all finite subsets are in $\mathcal{B}$ or for each $m \in \mathbb{N}$ there is a finite subset F of Z such that all m element subsets of $Z \setminus F$ are in $\mathcal{B}$. In our study of functions $f\colon \mathcal{F} \to Z$, we are seeking incompatibilities, that is, we are looking for large (i.e., with empty intersection) subsets of $\mathcal{F}$ that are mapped by f to small (i.e., belonging to $\mathcal{B}$) subsets of Z. The three main results of Section 2 (Theorems 2.1, 2.4 and 2.8) have this form. Theorem 2.1 is instrumental in the considerations of Section 3.

Convention. The symbol $\mathbb{N}$ stands for the set of natural numbers and it contains 0. We will sometimes identify $n \in \mathbb{N}$ with the initial segment determined by it, that is, with $\{i \in \mathbb{N} \colon i < n\}$.

2. Avoiding families

2.1. **Definitions and notation.** Recall from the introduction two main notions of this section. A family $\mathcal{F}$ of subsets of $2^{\mathbb{N}}$ is called *avoiding* if each nowhere dense subset of $2^{\mathbb{N}}$ is disjoint from an element of $\mathcal{F}$. We note that families that are not avoiding are closed under taking finite unions and taking subfamilies, that is, they form an ideal. For a set Z, $\mathcal{B}$ is called a *collective on* Z if $\mathcal{B}$ is a family of finite subsets of Z, $\bigcup \mathcal{B} = Z$, and $\mathcal{B}$ is closed under taking subsets.

The following notion comparing families of clopen sets will be relevant. For families $\mathcal{F}_1, \mathcal{F}_2$ of clopen sets, let $\mathcal{F}_2 \ll \mathcal{F}_1$ if for each $V \in \mathcal{F}_2$ there exists $U \in \mathcal{F}_1$ such that $U \subseteq V$.

We will now introduce some notions and notation concerning finite partial functions from $\mathbb{N}$ to $\{0,1\}$ and sequences of such functions. If s is a function from a finite subset of $\mathbb{N}$ to $\{0,1\}$, let

$$[s] = \{x \in 2^{\mathbb{N}} \colon s \subseteq x\}.$$

For a sequence $(s_0, \dots, s_{k-1})$ of functions from finite subsets of $\mathbb{N}$ to $\{0,1\}$, we put

$$](s_0, \dots, s_{k-1})[= \bigcup_{i<k} [s_i].$$

In particular, for the empty sequence we have $]\emptyset[= \emptyset$.

A family $\mathcal{G}$ of finite sequences of functions from finite subsets of $\mathbb{N}$ to $\{0,1\}$ is called *avoiding* if $\{]\bar{s}[\,:\, \bar{s} \in \mathcal{G}\}$ is avoiding. Note that $\mathcal{G} = \{\emptyset\}$ is avoiding.

Let $\mathcal{G}_{<\infty}$ be the collection of all finite sequences (possibly empty) $\bar{s} = (s_0, s_1, \dots, s_{k-1})$, for some natural number k, where each s_i, $i < k$, is a function from a subset of $\mathbb{N}$ to $\{0,1\}$ whose domain is a non-empty finite interval, and the sequence $\bar{s}$ is such that $\min \operatorname{dom}(s_0) = 0$, $\max \operatorname{dom}(s_{i-1}) + 1 = \min \operatorname{dom}(s_i)$ for $1 \leq i < k$. For $\bar{s} = (s_0, \dots, s_{k-1}) \in \mathcal{G}_{<\infty}$, we define

$$\operatorname{dom}(\bar{s}) = \operatorname{dom}(s_0) \cup \dots \cup \operatorname{dom}(s_{k-1}).$$

We say that the length of $\bar{s}$ as above is k, in symbols,

$$\operatorname{lh}(\bar{s}) = k.$$

Let $2^{<\mathbb{N}}$ stand for the set of all functions defined on a proper non-empty initial segment of $\mathbb{N}$. If $s \in 2^{<\mathbb{N}}$ and $\operatorname{dom}(s) = \{0, \dots, k-1\}$ for some $k \in \mathbb{N}$, we write

$$\operatorname{lh}(s) = k.$$

If $n \in \mathbb{N}$, let $s \restriction n = s \restriction \{0, \ldots, n-1\}$. In particular, if $n \geq \mathrm{lh}(s)$, then $s \restriction n = s$. For $s, t \in 2^{<\mathbb{N}}$, we write

$$t \perp s$$

if there is $i \in \mathbb{N}$ in the domain of both s and t with $s(i) \neq t(i)$. If $t \in 2^{<\mathbb{N}}$ and $\bar{s} = (s_0, \ldots, s_{k-1}) \in \mathcal{G}_{<\infty}$, we write

$$t \perp \bar{s}$$

if for each $i < k$ there exists $j \in \mathrm{dom}(t) \cap \mathrm{dom}(s_i)$ with $t(j) \neq s_i(j)$.

Let $\mathcal{G} \subseteq \mathcal{G}_{<\infty}$. For $s \in 2^{<\mathbb{N}}$ define

$$(\mathcal{G})_s = \{\bar{s} : (s)^\frown \bar{s} \in \mathcal{G}\}$$

A family $\mathcal{G} \subseteq \mathcal{G}_{<\infty}$ is called *well-founded* if for each sequence $(s_i)_{i\in\mathbb{N}}$ with $(s_i)_{i<n} \in \mathcal{G}_{<\infty}$ for each $n \in \mathbb{N}$ there exists n_0 such that the sequence $(s_i)_{i<n_0}$ cannot be extended to a sequence in $\mathcal{G}$. This notion of well-foundedness is directly related to the following tree associated with $\mathcal{G}$. Define

$$Tr(\mathcal{G}) = \{\bar{s} \restriction k : \bar{s} \in \mathcal{G},\ k \leq \mathrm{lh}(\bar{s})\}.$$

It is clear that with the relation of end-extension $Tr(\mathcal{G})$ is a tree and this tree is well-founded precisely when $\mathcal{G}$ is well-founded. We call $Tr(\mathcal{G})$ the *tree of* $\mathcal{G}$. If $Tr(\mathcal{G})$ is well-founded, there is the usual notion of an ordinal valued rank ρ associated with it [6, 2E]: for $\bar{s} \in \mathcal{G}_{<\infty}$ let

$$\rho(\bar{s}, Tr(\mathcal{G})) = \sup\{\rho(\bar{t}, Tr(\mathcal{G})) + 1 \colon \bar{t} \in Tr(\mathcal{G}),\ \bar{t} \supsetneq \bar{s}\},$$

with the convention that the supremum of the empty set is 0. For a well-founded non-empty $\mathcal{G}$, we let

$$rk(\mathcal{G}) = \rho(\emptyset, Tr(\mathcal{G})).$$

We call this countable ordinal the *rank of* $\mathcal{G}$. Note that $rk(\mathcal{G}) = 0$ precisely when $\mathcal{G} = \{\emptyset\}$.

The above rank will be used in proofs by induction. Since these arguments will involve an abuse of terminology, which frees the language from unnecessary complications, we say a few words about them. We will be inductively proving a statement for all avoiding well-founded families. We will be given an avoiding well-founded family $\mathcal{G} \subseteq \mathcal{G}_{<\infty}$ with rank α. We will find $s \in 2^{<\mathbb{N}}$ such that $(\mathcal{G})_s$ is also avoiding. Strictly speaking $(\mathcal{G})_s$ is not a subset of $\mathcal{G}_{<\infty}$, but the family that is the image of $(\mathcal{G})_s$ by the map induced by the unique order preserving bijection between $[\max \mathrm{dom}(s) + 1, \infty)$ and $\mathbb{N}$ is. This family is avoiding, well-founded, and its rank is less than α. So the inductive assumption can be applied to it. We will write our arguments as if the assumption were being applied to $(\mathcal{G})_s$ itself.

2.2. **Functions from avoiding families.** The theorem below will be crucial in proving the non-existence of a Tukey reduction from NWD to density-like ideals.

Theorem 2.1. *Let $\mathcal{F}_n$, $n \in \mathbb{N}$, be avoiding families of clopen sets. Let $f_n\colon \mathcal{F}_n \to Z$ for some set Z. Let $\mathcal{B}$ be a collective on Z such that for each infinite subset Y of Z there exists an infinite set $X \subseteq Y$ all of whose finite subsets are in $\mathcal{B}$. Then for each $n_0 \in \mathbb{N}$ there exist natural numbers $n_1 < \cdots < n_k$ with $n_0 < n_1$ and clopen sets $U_0 \in \mathcal{F}_{n_0}, \ldots, U_k \in \mathcal{F}_{n_k}$ such that*

(i) $\bigcap_{0\leq i\leq k} U_i = \emptyset$;
(ii) $\{f_{n_1}(U_1), \ldots, f_{n_k}(U_k)\} \in \mathcal{B}$.

We point out that in (ii) above the enumeration of the set starts with 1 not with 0; see Theorems 2.4 and 2.8.

Lemma 2.2. *If $\mathcal{F}$ is an avoiding family of clopen subsets of $2^{\mathbb{N}}$, then there exists an avoiding family $\mathcal{G} \subseteq \mathcal{G}_{<\infty}$ such that $\{\,]\bar{s}[\, : \bar{s} \in \mathcal{G}\} \ll \mathcal{F}$.*

Proof. Let $\mathcal{G}$ consist of all $\bar{s} \in \mathcal{G}_{<\infty}$ for which there exists $U \in \mathcal{F}$ with $U \subseteq\,]\bar{s}[$. It is clear that $\{\,]\bar{s}[\, : \bar{s} \in \mathcal{G}\} \ll \mathcal{F}$. We claim that $\{\,]\bar{s}[\, : \bar{s} \in \mathcal{G}\}$ is avoiding. Let F be nowhere dense. It is a standard observation that there exists an infinite sequence $(s_i)_{i\in\mathbb{N}}$ such that for each n, $(s_i)_{i<n} \in \mathcal{G}_{<\infty}$ and F is disjoint with

$$\{x \in 2^{\mathbb{N}} : \exists i\; s_i \subseteq x\}.$$

Since the set above is open and dense, there exists $U \in \mathcal{F}$ contained in it. Since U is compact, there exists n such that

$$U \subseteq \{x \in 2^{\mathbb{N}} : \exists i < n\; s_i \subseteq x\} = \,](s_i)_{i<n}[\,.$$

Since $(s_i)_{i<n} \in \mathcal{G}_{<\infty}$, it follows that $(s_i)_{i<n} \in \mathcal{G}$ and, obviously, $](s_i)_{i<n}[$ is disjoint from F. □

Lemma 2.3. *Let $\mathcal{G} \subseteq \mathcal{G}_{<\infty}$ be avoiding with $\emptyset \notin \mathcal{G}$. There exists $t_0 \in 2^{<\mathbb{N}}$ such that for each $s \supseteq t_0$ there exists $t \supseteq s$ with $(\mathcal{G})_t$ avoiding.*

Proof. Consider B defined to be the set of all $s \in 2^{<\mathbb{N}}$ such that $(\mathcal{G})_s$ is not avoiding. The non-avoidance of $(\mathcal{G})_s$ is witnessed by some nowhere dense set. This set can be easily modified so that, for each $s \in B$, we get a nowhere dense subset F_s of $2^{\mathbb{N}}$ such that for each $x \in F_s$, $x \restriction \mathrm{dom}(s)$ is constantly equal to 0 and $F_s \cap\,]\bar{s}[\, \neq \emptyset$ for each $\bar{s} \in (\mathcal{G})_s$. Then

$$F = \bigcup_{s\in B} F_s$$

is nowhere dense (as there are only finitely many s of length less than a given natural number) and intersects all $](s)^\frown\bar{s}[$ with $s \in B$ and with $(s)^\frown\bar{s} \in \mathcal{G}$.

Let $A = 2^{<\mathbb{N}} \setminus B$. It suffices to see that there is a t_0 such that each element of $2^{<\mathbb{N}}$ extending t_0 is extended by an element of A. Otherwise, each $t_0 \in 2^{<\mathbb{N}}$ can be extended to an element of

$$B' = \{t \in 2^{<\mathbb{N}} \colon \forall s \supseteq t\ s \in B\}.$$

Let B'' be the set of all $t \in B'$ that are minimal under inclusion sequences in B'. For $t \in B''$, pick $x_t \in [t]$. Then the set

$$F' = \{x_t \colon t \in B''\} \cup (2^{\mathbb{N}} \setminus \bigcup\{[t] \colon t \in B'\}) \tag{3}$$

is in NWD since, by our assumption, $\bigcup\{[t] \colon t \in B'\}$ is dense and, of course, open. The set F' intersects each $[s]$ with $s \in A$. Indeed, if $[s]$ does not intersect the second set in the union in (3), then

$$[s] \subseteq \bigcup\{[t] \colon t \in B'\}.$$

It follows from this inclusion and from $s \notin B$ that there is $t \in B''$ with $s \subseteq t$, so $x_t \in [s]$. Therefore, F' intersects all $](t)^\frown\bar{t}[$ for $t \in A$ and $(t)^\frown\bar{t} \in \mathcal{G}$.

The nowhere dense set $F \cup F'$ intersects all $]\bar{s}[$ with $\bar{s} \in \mathcal{G}$ and $\bar{s} \neq \emptyset$. Since we assume that $\emptyset \notin \mathcal{G}$, this statement contradicts our assumption that $\mathcal{G}$ is avoiding. □

Proof of Theorem 2.1. We are given a sequence $\mathcal{F}_n$, $n \in \mathbb{N}$, of avoiding families and functions f_n. Note first that if $\emptyset \in \mathcal{F}_n$ for infinitely many n, then we are done since for each such n, $\{f_n(\emptyset)\} \in \mathcal{B}$ as $\mathcal{B}$ contains all singletons of Z. Therefore, we can assume that $\emptyset \notin \mathcal{F}_n$ for all n.

First we use Lemma 2.2 to find avoiding families $\mathcal{G}_n \subseteq \mathcal{G}_{<\infty}$ such that

$$\{\,]\bar{s}[\, \colon \bar{s} \in \mathcal{G}_n\} \ll \mathcal{F}_n. \tag{4}$$

Note that $\emptyset \notin \mathcal{G}_n$ for each n. Define $f^n \colon \mathcal{G}_n \to Z$ by letting $f^n(\bar{s})$ be equal to $f_n(U)$ for some $U \in \mathcal{G}_n$ with $U \subseteq]\bar{s}[$. Note that such a U exists by (4). Note also that, given n_0, it will suffice to find $n_1 < \cdots < n_k$ with $n_0 < n_1$ and $\bar{s}_0 \in \mathcal{G}_{n_0}, \ldots, \bar{s}_k \in \mathcal{G}_{n_k}$ so that

$$\bigcap_{i=0}^{k}]\bar{s}_i[= \emptyset \quad \text{and} \quad \{f^{n_1}(\bar{s}_1), \ldots, f^{n_k}(\bar{s}_k)\} \in \mathcal{B}.$$

By Lemma 2.3, we can find $t_n \in 2^{<\mathbb{N}}$ such that for a dense set of t extending t_n we have that $(\mathcal{G}_n)_t$ is avoiding. We can assume that

$$\mathrm{lh}(t_n) \to \infty \text{ as } n \to \infty. \tag{5}$$

By going to a subsequence of (t_n), we can also assume that there exists $x_0 \in 2^{\mathbb{N}}$ such that for each $s \subseteq x_0$ we have $s \subseteq t_n$ for large enough n. Now by extending t_n to some sequence in $2^{<\mathbb{N}}$, which we will again call t_n, we can guarantee that

$$t_n \not\subseteq x_0 \tag{6}$$

and that $(\mathcal{G}_1)_{t_n}$ remains avoiding. By this last condition, we can pick $\bar{t}_n \in (\mathcal{G}_1)_{t_n}$ such that

$$x_0 \notin \,]\bar{t}_n[. \tag{7}$$

We now use (5) to go to a subsequence of (t_n) and then re-enumerate it so that for each n we have

$$\max\{\mathrm{lh}(s)\colon s \subseteq t_{n+1} \text{ and } s \subseteq x_0\} > \max \mathrm{dom}((t_n)^\frown \bar{t}_n).$$

The equation above together with (6) and (7) imply that for $p > n$

$$]\,(t_n)^\frown \bar{t}_n[\, \cap \{x \in 2^{\mathbb{N}}\colon t_p \subseteq x\} = \emptyset. \tag{8}$$

It follows from (8) that for each sequence $n_1 < n_2 < \cdots$ we have

$$\bigcap_{k=1}^{\infty} \,]\,(t_{n_k})^\frown \bar{t}_{n_k}[\, = \,]\,(t_{n_1})^\frown \bar{t}_{n_1}[\, \cap \bigcap_{k\geq 2} \,]\bar{t}_{n_k}[\, .$$

Since by (5) the set $\bigcap_{k\geq 2}]\bar{t}_{n_k}[$ is nowhere dense, the equality above implies that for each sequence $n_1 < n_2 < \cdots$ we have

$$\bigcap_{k=1}^{\infty} \,]\,(t_{n_k})^\frown \bar{t}_{n_k}[\, \in \mathrm{NWD}. \tag{9}$$

Consider the sequence $(f^n((t_n)^\frown \bar{t}_n))_n$ of elements of Z. If the range of the sequence is finite, there is one value that is taken infinitely many times. Since $\mathcal{B}$ is a collective on Z, the singleton consisting of this one value is in $\mathcal{B}$. If the range of the sequence is infinite, we apply to it our assumption on $\mathcal{B}$. In either case, we can pick $n_1 < n_2 < \cdots$ so that $n_0 < n_1$ and for each k we have

$$\{f^{n_1}(t_{n_1}^\frown \bar{t}_{n_1}), \ldots, f^{n_k}(t_{n_k}^\frown \bar{t}_{n_k})\} \in \mathcal{B}. \tag{10}$$

For this subsequence we also have (9). Since $\mathcal{G}_{n_0}$ is avoiding, property (9) allows us to find $\bar{s}_0 \in \mathcal{G}_{n_0}$ so that

$$]\bar{s}_0[\, \cap \bigcap_{k=1}^{\infty} \,]t_{n_k}^\frown \bar{t}_{n_k}[\, = \emptyset.$$

By compactness, we can find k_0 with

$$]\bar{s}_0[\, \cap \bigcap_{k=1}^{k_0} \,]t_{n_k}^\frown \bar{t}_{n_k}[\, = \emptyset. \tag{11}$$

The theorem follows from (11) and (10) since $\bar{s}_0 \in \mathcal{G}_{n_0}$, $t_{n_k}^\frown \bar{t}_{n_k} \in \mathcal{G}_{n_k}$, for $1 \leq k \leq k_0$, and $n_0 < n_1 < \cdots < n_{k_0}$. □

One of the more interesting collectives $\mathcal{B}$ as in Theorem 2.1 is the family $\mathcal{B}_0$ consisting of those finite subsets F of the set $Z = 2^{<\mathbb{N}}$ for which we have, for each $l \in \mathbb{N}$,

$$|\{s \restriction l : s \in F \text{ and } l \subseteq \mathrm{dom}(s)\}| \leq 2.$$

We will let the reader check that this family fulfills the condition from the assumption of Theorem 2.1. The following theorem shows that for the family $\mathcal{B}_0$ we can do better than in point (ii) of Theorem 2.1 in the case when the avoiding families $\mathcal{F}_n$ and the functions f_n are equal for different n.

Theorem 2.4. *Let $\mathcal{F}$ be an avoiding family of clopen subsets of $2^{\mathbb{N}}$ and let $f\colon \mathcal{F} \to 2^{<\mathbb{N}}$. There exist $U_0, \ldots, U_k \in \mathcal{F}$ such that*

(i) $\bigcap_{i \leq k} U_i = \emptyset$;
(ii) $\{f(U_0), \ldots, f(U_k)\} \in \mathcal{B}_0$.

Note that in point (ii) above the sequence starts at 0 and not at 1 as in Theorem 2.1(ii).

The following lemma strengthens Lemma 2.2. Recall the definition of well-fundedness of a family $\mathcal{G} \subseteq \mathcal{G}_{<\infty}$ from Subsection 2.1.

Lemma 2.5. *If $\mathcal{F}$ is an avoiding family of clopen subsets of $2^{\mathbb{N}}$, then there exists an avoiding family $\mathcal{G} \subseteq \mathcal{G}_{<\infty}$ such that $\mathcal{G}$ is well-founded and $\{\,]\bar{s}[\, : \bar{s} \in \mathcal{G}\} \ll \mathcal{F}$.*

Proof. By Lemma 2.2, it will suffice to show the following: given an avoiding family $\mathcal{G}' \subseteq \mathcal{G}_{<\infty}$, there exists an avoiding family $\mathcal{G} \subseteq \mathcal{G}'$ for which $Tr(\mathcal{G})$ is well-founded. For $\bar{s} = (s_0, \ldots, s_{m-1}), \bar{t} = (t_0, \ldots, t_{n-1}) \in \mathcal{G}_{<\infty}$, we write $\bar{s} \prec \bar{t}$ if the following condition holds: $m < n$ and for each $i < m$ there exists $j < n$ such that $t_j \subseteq s_i$. We leave it to the reader to check that $\prec$ is a strict partial ordering that is well-founded. Now, let $\mathcal{G}$ consist of all elements of $\mathcal{G}'$ that are $\prec$-minimal in $\mathcal{G}'$.

The family $\mathcal{G}$ is avoiding since for any $\bar{t} \in \mathcal{G}' \setminus \mathcal{G}$ there exists $\bar{s} \in \mathcal{G}$ with $\bar{s} \prec \bar{t}$, and hence with $]\bar{s}[\, \subseteq \,]\bar{t}[$. To see that $\mathcal{G}$ is well-founded, assume towards a contradiction that it is not and consider an infinite sequence $(t_j)_{j \in \mathbb{N}}$ witnessing it. The set

$$\{x \in 2^{\mathbb{N}} : \exists j \; t_j \subseteq x\}$$

is open and dense. Since $\mathcal{G}$ is avoiding, there is $(s_0, \ldots, s_{m-1}) \in \mathcal{G}$ such that $](s_0, \ldots, s_{m-1})[$ is contained in this set. It is easy to see that this containment implies that for each $i < m$ there exists j_i such that $t_{j_i} \subseteq s_i$. Let n be such that $n > j_i$, for each $i < m$, and $n > m$. (In fact, the first one of these conditions implies the second one.) By the choice of $(t_j)_{j \in \mathbb{N}}$, there exists $\bar{t}' \in \mathcal{G}$ such that $\bar{t}' \upharpoonright n = (t_j)_{j<n}$. It follows immediately that $\bar{s} \prec \bar{t}'$ and both these sequences are in $\mathcal{G}$, which contradicts the definition of $\mathcal{G}$. $\square$

Recall the notion of $t \perp \bar{s}$ for $t \in 2^{<\mathbb{N}}$ and $\bar{s} \in \mathcal{G}_{<\infty}$ from Subsection 2.1. We would like to be a little more precise about point (i) in Theorem 2.4. To this end we introduce the following notion. A subset of $\mathcal{G}_{<\infty}$ is called a *spherical configuration* if it belongs to the smallest family of subsets of $\mathcal{G}_{<\infty}$ that

1. contains the set $\mathcal{C}$ whose only element is the empty sequence and
2. is closed under the following operation: from $\bar{s} \in \mathcal{G}_{<\infty}$, $t \in 2^{<\mathbb{N}}$ such that $t \perp \bar{s}$, and $\mathcal{C}'$ a spherical configuration on $[(\max \mathrm{dom}(t)) + 1, \infty)$, we produce

$$\mathcal{C} = \{\bar{s}\} \cup \{(t)^\frown \bar{t} : \bar{t} \in \mathcal{C}'\}.$$

Above, by *$\mathcal{C}'$ is a spherical configuration on $[n, \infty)$* we mean that the image of $\mathcal{C}'$ in $\mathcal{G}_{<\infty}$ by the natural function induced by the unique order preserving bijection between $[n, \infty)$ and $\mathbb{N}$ is a spherical configuration.

Note that spherical configurations are always non-empty.

Lemma 2.6. *Let $\mathcal{C}$ be a spherical configuration. Then*

$$\bigcap \{\,]\bar{s}[\, : \bar{s} \in \mathcal{C}\} = \emptyset.$$

Proof. The proof is by recursion. Note first that if $\mathcal{C}'$ is a spherical configuration on $[n, \infty)$, then its intersection is empty precisely when the intersection of the image of $\mathcal{C}'$ in $\mathcal{G}_{<\infty}$ by the natural function induced by the unique order preserving bijection between $[n, \infty)$ and $\mathbb{N}$ is empty.

If $\mathcal{C} = \{\emptyset\}$, then

$$\bigcap \{\,]\bar{s}[\, : \bar{s} \in \mathcal{C}\} = \,]\emptyset[\, = \emptyset.$$

Let $\mathcal{C} = \{\bar{s}'\} \cup \{(t')^\frown \bar{t}' : \bar{t}' \in \mathcal{C}'\}$ be obtained from appropriately chosen $\bar{s}'$ and t' and a spherical configuration $\mathcal{C}'$ on $[(\max \mathrm{dom}(t')) + 1, \infty)$ with

$$\bigcap_{\bar{t}' \in \mathcal{C}'} \,]\bar{t}'[\, = \emptyset \tag{12}$$

using the operation described in point 2 of the definition of spherical configurations. Assume towards a contradiction that $x \in \bigcap_{\bar{s} \in \mathcal{C}}]\bar{s}[$. Then $x \in \,]\bar{s}'[$, hence $t' \not\subseteq x$. Thus, $x \in \bigcap_{\bar{t}' \in \mathcal{C}'}]\bar{t}'[$ contradicting (12). $\square$

Lemma 2.7. *Let $\mathcal{G} \subseteq \mathcal{G}_{<\infty}$ be avoiding and well-founded. Let $f : \mathcal{G} \to 2^{<\mathbb{N}}$. Then for each $n \in \mathbb{N}$ there is a spherical configuration $\mathcal{C} \subseteq \mathcal{G}$ such that*

(i) *$f(\bar{s}) \upharpoonright n = f(\bar{t}) \upharpoonright n$ for $\bar{s}, \bar{t} \in \mathcal{C}$;*
(ii) *for each $l \in \mathbb{N}$,*

$$|\{f(\bar{s}) \upharpoonright l : \bar{s} \in \mathcal{C},\ l \subseteq \mathrm{dom}(f(\bar{s}))\}| \leq 2.$$

Proof. The proof is by induction on the rank of $\mathcal{G}$. (Recall the definition of rank of $\mathcal{G}$ from Subsection 2.1.) If the rank is 0, then $\mathcal{G}$ consist of the empty sequence, and we take $\mathcal{C} = \{\emptyset\}$.

Assume that α is a countable ordinal and the conclusion holds for all $\mathcal{G}$ with rank strictly less than α. Let now $\mathcal{G}$ be avoiding, well-founded and of rank α, and let $n \in \mathbb{N}$. If $\emptyset \in \mathcal{G}$, we can take again $\mathcal{C} = \{\emptyset\}$. So assume $\emptyset \notin \mathcal{G}$. We fix t_0 given by Lemma 2.3 applied to $\mathcal{G}$. By recursion on $i \in \mathbb{N}$, we construct s_i, $\bar{s}_i$, $\mathcal{C}_i$ so that

(i) $s_i \in 2^{<\mathbb{N}}$ and $t_0 \subsetneq s_i$;
(ii) $\mathcal{C}_i$ is a spherical configuration on $[(\max \operatorname{dom}(s_i)) + 1, \infty)$ included in $(\mathcal{G})_{s_i}$;
(iii) $\bar{s}_i = (s_i)^\frown \bar{t}$ for some $\bar{t} \in \mathcal{C}_i$;
(iv) $\operatorname{dom}(s_i) \supsetneq \operatorname{dom}(\bar{s}_{i-1})$, $s_{i-1} \restriction (\operatorname{lh}(s_{i-1}) - 1) \subseteq s_i$, and $s_i \perp \bar{s}_{i-1}$ if $i > 0$;
(v) $f((s_i)^\frown \bar{t}) \restriction n = f((s_i)^\frown \bar{t}') \restriction n$ for all $\bar{t}, \bar{t}' \in \mathcal{C}_i$;
(vi) $f((s_i)^\frown \bar{t}) \restriction m_i = f((s_i)^\frown \bar{t}') \restriction m_i$ for all $\bar{t}, \bar{t}' \in \mathcal{C}_i$, where m_i is set to be $\max_{j<i} \operatorname{dom}(f(\bar{s}_j))$;
(vii) for each $l \in \mathbb{N}$,

$$|\{f((s_i)^\frown \bar{t}) \restriction l : \bar{t} \in \mathcal{C}_i,\, l \subseteq \operatorname{dom}(f((s_i)^\frown \bar{t}))\}| \leq 2.$$

Note that in point (vi), $m_0 = 0$.

Assume that the construction has been carried out. By (v) and the pigeonhole principle, there exist $i < j$ such that

$$\forall \bar{t} \in \mathcal{C}_i, \bar{t}' \in \mathcal{C}_j \ (f((s_i)^\frown \bar{t}) \restriction n = f((s_j)^\frown \bar{t}') \restriction n).$$

With these i, j, we define

$$\mathcal{C} = \{\bar{s}_i\} \cup \{(s_j)^\frown \bar{t} : \bar{t} \in \mathcal{C}_j\}.$$

Note that (iv) implies that $s_j \perp \bar{s}_i$. Therefore, by point (ii), $\mathcal{C}$ is a spherical configuration in $\mathcal{G}$. By (iii), (v), and our choice of i and j, Lemma 2.7(i) follows. Finally, by (vi), and (vii), we get Lemma 2.7(ii).

Now we describe the construction of s_i, $\bar{s}_i$, and $\mathcal{C}_i$. For the sake of convenience, set $s_{-1} \in 2^{<\mathbb{N}}$ to be a proper extension of t_0, and let $\bar{s}_{-1} = (s_{-1})$ be the sequence with a single entry equal to s_{-1}. Assume now that s_i and $\bar{s}_i$ have been produced. We construct these objects for $i+1$ along with $\mathcal{C}_{i+1}$. First let $t \in 2^{<\mathbb{N}}$ be such that $\operatorname{dom}(t)$ properly contains $\operatorname{dom}(\bar{s}_i)$, $s_i \restriction (\operatorname{lh}(s_i) - 1) \subseteq t$ and $t \perp \bar{s}_i$. By (i), we have $t \supseteq t_0$. It follows that there exists $s_{i+1} \supseteq t$ with $(\mathcal{G})_{s_{i+1}}$ avoiding. We see right away that points (i) and (iv) hold for $i+1$. Obviously the three of $(\mathcal{G})_{s_{i+1}}$ is well-founded and has rank $< \alpha$. Let

$$n' = \max\bigl(n, \max_{j<i} \operatorname{dom}(f(\bar{s}_j))\bigr).$$

By inductive assumption on α, there exists a spherical configuration $\mathcal{C}_{i+1} \subseteq (\mathcal{G})_{s_{i+1}}$ such that

— $f((s_{i+1})^\frown \bar{t}) \restriction n' = f((s_{i+1})^\frown \bar{t}') \restriction n'$ for all $\bar{t}, \bar{t}' \in \mathcal{C}_{i+1}$ and

— for each $l \in \mathbb{N}$,

$$|\{f((s_{i+1})^\frown \bar{t}) \restriction l : \bar{t} \in \mathcal{C}_{i+1},\, l \subseteq \mathrm{dom}(f((s_{i+1})^\frown \bar{t}))\}| \leq 2.$$

We, of course, have (ii) for $i+1$ and the conditions above ensure that (v), (vi), and (vii) hold, as well. Pick some $\bar{t} \in \mathcal{C}_{i+1}$, and let $\bar{s}_{i+1} = (s_{i+1})^\frown \bar{t}$, which makes point (iii) true. □

Proof of Theorem 2.4. Let $\mathcal{F}$ be an avoiding family and let $f : \mathcal{F} \to 2^{<\mathbb{N}}$. By Lemma 2.5, find $\mathcal{G} \subseteq \mathcal{G}_{<\infty}$ such that $\mathcal{G}$ is avoiding, well-founded, and

$$\{\,]\bar{s}[\,:\, \bar{s} \in \mathcal{G}\} \ll \mathcal{F}.$$

This last condition allows us to find for each $\bar{s} \in \mathcal{G}$ a set $U_{\bar{s}} \in \mathcal{F}$ with

$$U_{\bar{s}} \subseteq \,]\bar{s}[. \tag{13}$$

Define $g : \mathcal{G} \to 2^{\mathbb{N}}$ by

$$g(\bar{s}) = f(U_{\bar{s}}).$$

By Lemma 2.7, there is a spherical configuration $\mathcal{C} \subseteq \mathcal{G}$ such that for each $l \in \mathbb{N}$,

$$|\{g(\bar{s}) \restriction l : \bar{s} \in \mathcal{C},\ l \subseteq \mathrm{dom}(g(\bar{s}))\}| \leq 2. \tag{14}$$

Now Lemma 2.6 together with relation (13) give that

$$\bigcap_{\bar{s} \in \mathcal{C}} U_{\bar{s}} = \emptyset.$$

This equality along with the definition of g and inequality (14) yield the conclusion of the theorem. □

The existence, for each k, of an avoiding family with the property that each k sets in it have non-empty intersection is crucial in proving the existence of Tukey reductions from NWD to ℓ_1 [3] or more broadly to summable-like ideals as in Theorem 3.7(i). Below in Corollary 2.10, we show that this last property cannot be improved, that is, in each "tail" of an avoiding family, one can find finite subfamilies of a fixed size with empty intersections. We will deduce this corollary from the following stronger theorem.

Theorem 2.8. *Let $\mathcal{F}$ be an avoiding family of clopen subsets of $2^{\mathbb{N}}$ and let $f \colon \mathcal{F} \to Z$. Let $\mathcal{B}$ be a collective on Z with the property that for each m there exists a finite set $F \subseteq Z$ such that all m element subsets of $Z \setminus F$ are in $\mathcal{B}$. Then there are $U_0, \ldots, U_k$ in $\mathcal{F}$ such that*

(i) $\bigcap_{i \leq k} U_i = \emptyset$;

(ii) $\{f(U_0), \ldots, f(U_k)\} \in \mathcal{B}$.

First we show that a statement slightly stronger than the one in the theorem above holds for avoiding and well-founded $\mathcal{G} \subseteq \mathcal{G}_{<\infty}$.

Lemma 2.9. *Let $\mathcal{G} \subseteq \mathcal{G}_{<\infty}$ be avoiding and well-founded. Let $g\colon \mathcal{G} \to Z$ for some set Z. Then either there exists finite $F \subseteq Z$ with $g^{-1}(Z \setminus F)$ not avoiding or there exists $m \in \mathbb{N}$ with the property that for each finite $F \subseteq Z$ there exist $\bar{s}_0, \dots, \bar{s}_k \in \mathcal{F}$ such that*

(i) $\bigcap_{i\leq k}]\bar{s}_i[= \emptyset$;
(ii) $\{g(]\bar{s}_0[), \dots, g(]\bar{s}_k[)\}$ *has at most m elements and is included in $Z \setminus F$.*

Proof. It will be convenient to introduce the following notion. We say that g is *condensed on* $F \subseteq Z$ if $g^{-1}(Z \setminus F)$ is not avoiding. Note that since non-avoiding sets form an ideal, in the case when g is condensed on a finite set, there exists a smallest, i.e., included in every other such set, finite set on which it is condensed.

We continue with the proof. If the rank of $\mathcal{G}$ is 0, then $\mathcal{G}$ consists of the empty sequence only and then $g^{-1}(Z \setminus \{g(\emptyset)\})$ is not avoiding since it is empty. Assume the theorem holds for all ranks less than $\alpha > 0$ and let $\mathcal{G}$ have rank α. If $\mathcal{G} \setminus \{\emptyset\}$ is non-avoiding, then the first alternative of the conclusion of the lemma holds with $F = \{g(\emptyset)\}$. If $\mathcal{G} \setminus \{\emptyset\}$ is avoiding, it is easy to see that it suffices to prove the lemma for $\mathcal{G} \setminus \{\emptyset\}$. So assume that $\emptyset \notin \mathcal{G}$.

For $t \in 2^{<\mathbb{N}}$ define $g_t\colon (\mathcal{G})_t \to Z$ by

$$g_t(\bar{s}) = g((t)^\frown \bar{s}).$$

Note that by the recursive assumption, if $(\mathcal{G})_t$ is avoiding, then the lemma holds for g_t. Define

$$A_0 = \{t \in 2^{<\mathbb{N}} \colon g_t \text{ is not condensed on a finite set}\}$$

$$A_1 = \{t \in 2^{<\mathbb{N}} \colon g_t \text{ is condensed on a finite set}\}.$$

For $t \in A_1$, let $F(t) \subseteq Z$ be the smallest (finite) set on which g_t is condensed. Note that if $(\mathcal{G})_t$ is not avoiding, then clearly $F(t) = \emptyset$ and that $t \in A_0$ implies that $(\mathcal{G})_t$ is avoiding. Note also that

$$A_0 \cup A_1 = 2^{<\mathbb{N}}.$$

Claim 1. If there are $t_0, t_1 \in A_0$ with $t_0 \perp t_1$, then the lemma holds for g.

Proof of Claim 1. The assumptions imply that $(\mathcal{G})_{t_0}$ and $(\mathcal{G})_{t_1}$ are both avoiding, their trees have rank less than α and g_{t_0} and g_{t_1} are not condensed on finite sets. Thus, we can find $m_0 \in \mathbb{N}$ for $(\mathcal{G})_{t_0}$ and $m_1 \in \mathbb{N}$ for $(\mathcal{G})_{t_1}$ as in the statement of the lemma. We claim that $m_0 + m_1$ works for $\mathcal{G}$. Indeed, pick $F \subseteq Z$ finite. Let

$$\bar{s}^0_0, \dots, \bar{s}^0_{k_0} \in g^{-1}_{t_0}(Z \setminus F) \text{ and } \bar{s}^1_0, \dots, \bar{s}^1_{k_1} \in g^{-1}_{t_1}(Z \setminus F)$$

be as given by the inductive assumption. Then clearly

$$(t_0)^\frown \bar{s}^0_0, \dots, (t_0)^\frown \bar{s}^0_{k_0}, (t_1)^\frown \bar{s}^1_0, \dots, (t_1)^\frown \bar{s}^1_{k_1}$$

is a sequence contained in $g^{-1}(Z \setminus F)$ whose image under g has at most $m_0 + m_1$ elements. Also the sequence is such that

$$\bigcap_{k \leq k_0}](t_0)^\frown \bar{s}^0_k[\cap \bigcap_{k \leq k_1}](t_1)^\frown \bar{s}^1_k[= \emptyset.$$

Indeed, if a point of $2^{\mathbb{N}}$ belongs to $\bigcap_{k \leq k_0}](t_0)^\frown \bar{s}^0_k[$, then it belongs to $[t_0]$ by the choice of $\bar{s}^0_0, \dots, \bar{s}^0_{k_0}$. Similarly if a point belongs to $\bigcap_{k \leq k_1}](t_1)^\frown \bar{s}^1_k[$, then it belongs to $[t_1]$. But since $t_0 \perp t_1$, no point belongs to both $[t_0]$ and $[t_1]$. □

Claim 2. If for each finite $F \subseteq Z$ there are $t_0, t_1 \in A_1$ with $t_0 \perp t_1$ and $F(t_0) \setminus F \neq \emptyset \neq F(t_1) \setminus F$, then the lemma holds for g.

Proof of Claim 2. Let $F \subseteq Z$ be finite. Let $t_0, t_1 \in A_1$ and $z_0, z_1 \in Z$ be such that $t_0 \perp t_1$, $z_0 \in F(t_0) \setminus F$, and $z_1 \in F(t_1) \setminus F$. By the choice of z_0 and z_1 and the fact that non-avoiding sets form an ideal, $g^{-1}_{t_0}(z_0)$ and $g^{-1}_{t_1}(z_1)$ are both avoiding. Thus, we can pick

$$\bar{s}^0_0, \dots, \bar{s}^0_{k_0} \in g^{-1}_{t_0}(z_0) \text{ and } \bar{s}^1_0, \dots, \bar{s}^1_{k_1} \in g^{-1}_{t_1}(z_1)$$

so that

$$\bigcap_{i \leq k_0}]\bar{s}^0_i[= \emptyset \quad \text{and} \quad \bigcap_{i \leq k_1}]\bar{s}^1_i[= \emptyset.$$

Then clearly the image under g of the sequence

$$(t_0)^\frown \bar{s}^0_0, \dots, (t_0)^\frown \bar{s}^0_{k_0}, (t_1)^\frown \bar{s}^1_0, \dots, (t_1)^\frown \bar{s}^1_{k_1}$$

is $\{z_0, z_1\}$. Also, by a simple argument as in Claim 1, the sequence is such that

$$\bigcap_{k \leq k_0}](t_0)^\frown \bar{s}^0_k[\cap \bigcap_{k \leq k_1}](t_1)^\frown \bar{s}^1_k[= \emptyset.$$

Thus, the statement of the lemma holds with $m = 2$. □

It will suffice to consider the situation not covered by the two claims. Assume therefore that the assumptions of both claims fail. We can then pick $x_0 \in 2^{\mathbb{N}}$, $x_1 \in 2^{\mathbb{N}}$, and a finite set $F \subseteq Z$ such that for each $t \in A_0$ we have $t \subseteq x_0$ and for each $t \in A_1$, either $t \subseteq x_1$ or $F(t) \subseteq F$. It follows that for each $t \in 2^{<\mathbb{N}}$ we have

$$\left(g^{-1}(Z \setminus F) \setminus \mathcal{G}'\right)_t \text{ is not avoiding,} \tag{15}$$

where

$$\mathcal{G}' = \{(t_0, \dots, t_m) \in \mathcal{G} \colon t_0 \subseteq x_0 \text{ or } t_0 \subseteq x_1\}.$$

Indeed, (15) is clear if $t \subseteq x_0$, so if $t \in A_0$, or if $t \subseteq x_1$. The remaining case is $t \in A_1$ and $t \nsubseteq x_1$. But then

$$\big(g^{-1}(Z \setminus F) \setminus \mathcal{G}'\big)_t \subseteq (g^{-1}(Z \setminus F))_t \subseteq g_t^{-1}(Z \setminus F(t)),$$

and (15) follows.

From (15) and from $\emptyset \notin \mathcal{G}$, by Lemma 2.3, we get that $g^{-1}(Z \setminus F) \setminus \mathcal{G}'$ is not avoiding. Note that $\mathcal{G}'$ is not avoiding either as witnessed by the nowhere dense set $\{x_0, x_1\}$. Thus, $g^{-1}(Z \setminus F)$ is not avoiding. It follows that g is condensed on the finite set F, and the lemma is proved. □

Proof of Theorem 2.8. By Lemma 2.5, we can find an avoiding, well-founded family $\mathcal{G} \subseteq \mathcal{G}_{<\infty}$ with

$$\{\,]\bar{t}[: \bar{t} \in \mathcal{G}\} \ll \mathcal{F}. \tag{16}$$

If we now define $g \colon \mathcal{G} \to Z$ by letting $g(\bar{t})$ be equal to $f(U)$ for some $U \in \mathcal{F}$ with $U \subseteq]\bar{t}[$, then by Lemma 2.9 either there is a finite set $F \subseteq Z$ with $g^{-1}(Z \setminus F)$ non-avoiding or there exists $m \in \mathbb{N}$ such that, for each finite $F \subseteq Z$, there are sequences $\bar{t}_0, \ldots, \bar{t}_k \in \mathcal{G}$ with $\bigcap_{i\leq k}]\bar{t}_i[= \emptyset$ and with the set $\{g(\bar{t}_i) \colon i \leq k\}$ having at most m elements and being included in $Z \setminus F$. If the second alternative holds, it also holds for f, and therefore the theorem follows. If the first alternative holds, then $g^{-1}(F)$ is avoiding and hence, by definition of g, $f^{-1}(F)$ is avoiding. Since the family of non-avoiding sets is closed under taking finite unions, it follows in that case that there exists $z \in F$ with $f^{-1}(z)$ avoiding. But then

$$\bigcap f^{-1}(z) = \emptyset$$

and compactness allows us to find sets $U_0, \ldots, U_k \in f^{-1}(z)$ with (i). Point (ii) holds as well since $\{z\} \in \mathcal{B}$ as $\mathcal{B}$, being a collective, contains all singletons of Z. □

Corollary 2.10. *Let $\mathcal{F}$ be an avoiding family of non-empty clopen subsets of $2^{\mathbb{N}}$. Then there exists m such that for each finite set $F \subseteq \mathcal{F}$ there are $U_0, \ldots, U_m \in \mathcal{F} \setminus F$ with $\bigcap_{i\leq m} U_i = \emptyset$.*

Proof. If the conclusion of the corollary fails, then for each $m \in \mathbb{N}$ there exists a finite set $F_m \subseteq \mathcal{F}$ such that for all $U_0, \ldots, U_m \in \mathcal{F} \setminus F_m$ we have $\bigcap_{i\leq m} U_i \neq \emptyset$. Note that since all sets in $\mathcal{F}$ are assumed non-empty, we can take $F_0 = \emptyset$. Let $Z = \mathcal{F}$ and consider the identity function $f \colon \mathcal{F} \to Z$. Let $\mathcal{B}$ be the collective on $Z = \mathcal{F}$ consisting of all $m+1$ element subsets of $\mathcal{F} \setminus F_m$ for each m. Now an application of Theorem 2.8 gives us a contradiction. □

3. Tukey functions from NWD to analytic P-ideals

3.1. **Analytic P-ideals.** For finite subsets x, y of $\mathbb{N}$, we write $x < y$ if $\max x < \min y$, with the convention that $\emptyset < x < \emptyset$ for each x.

Let I be an analytic P-ideal of subsets of $\mathbb{N}$, that is, we assume that for each sequence $x_n \in I$, $n \in \mathbb{N}$, there is $x \in I$ with $x_n \setminus x$ finite for each n. By convention we have that $\{k\} \in I$ for each $k \in \mathbb{N}$. By [12], we can fix a lower semicontinous submeasure $\phi : 2^{\mathbb{N}} \to \mathbb{R}$ with

$$I = \mathrm{Exh}(\phi) = \{x \subseteq \mathbb{N} \colon \lim_n \phi(x \setminus \{0, \dots, n-1\}) = 0\}.$$

We can always assume, and we do, that $\phi(\{n\}) > 0$ for each $n \in \mathbb{N}$. For $x, y \in I$, define

$$d_\phi(x, y) = \phi(x \triangle y). \tag{17}$$

It can be checked that d_ϕ is a complete separable metric on I; see [12]. We call the topology induced by it on I the *submeasure topology.*

The following notion already played a role in [14]. We call an analytic P-ideal $I = \mathrm{Exh}(\phi)$ *density-like* if for each $\epsilon > 0$ there exists $\delta > 0$ such that for each sequence (x_n) of finite subsets of $\mathbb{N}$ with $\phi(x_n) < \delta$ and $x_n < x_{n+1}$ for each n, there exist $n_0 < n_1 < \cdots$ such that

$$\phi(\bigcup_k x_{n_k}) < \epsilon.$$

This property does not depend on the choice of ϕ representing I as $\mathrm{Exh}(\phi)$, since one quite easily sees that, for two such submeasures ϕ' and ϕ'', for each real number $r' > 0$ there is $r'' > 0$ such that, for all $x \subseteq \mathbb{N}$, $\phi''(x) < r''$ implies $\phi'(x) < r'$. It is even easier to see that $\mathcal{Z}_0$ is density-like as it is equal to $\mathrm{Exh}(\phi_0)$ where

$$\phi_0(x) = \sup_n \frac{|x \cap \{0, \dots, n\}|}{n+1}.$$

Also we see directly from their definitions that the analytic P-ideals considered in [7, Section 4] are density-like. On the other hand, F_σ P-ideals, for example ℓ_1, are not density-like, see [14, Section 6.2].

Lemma 3.1. *Let I be an analytic P-ideal. The following conditions are equivalent*

(i) *I is density-like;*
(ii) *for each $\epsilon > 0$ there exists $\delta > 0$ such that for each sequence (x_n) of sets in I with $\phi(x_n) < \delta$ for each n, there exist $n_0 < n_1 < \cdots$ with*

$$\phi(\bigcup_k x_{n_k}) < \epsilon.$$

Proof. Since finite subsets of $\mathbb{N}$ are in I, (ii) implies (i). To see that (i) implies (ii), fix $\epsilon > 0$. For $\epsilon/3$ fix $\delta > 0$ as given by (i). We can assume that $\delta \leq \epsilon/3$. Let (x_n) be a sequence of sets in I with $\phi(x_n) < \delta$ for all n. By going to a subsequence we can assume that $x_n \to x$ in $2^{\mathbb{N}}$ for some x. By lower semicontinuity of ϕ we have $\phi(x) \leq \delta$. Let $y_n \subseteq x_n \setminus x$ be finite and such that

$$\sum_n \phi\left(x_n \setminus (x \cup y_n)\right) < \frac{\epsilon}{3}.$$

By going to a subsequence, we can assume that $y_n < y_{n+1}$ for each n. By (i), there are $n_0 < n_1 < \cdots$ such that $\phi(\bigcup_i y_{n_i}) < \epsilon/3$. It follows that

$$\begin{aligned}\phi(\bigcup_i x_{n_i}) &\leq \phi(\bigcup_i y_{n_i}) + \phi(x) + \phi\left(\bigcup_i x_{n_i} \setminus (x \cup y_{n_i})\right) \\ &< \frac{\epsilon}{3} + \delta + \frac{\epsilon}{3} < \epsilon,\end{aligned}$$

as required. □

We call an analytic P-ideal $I = \mathrm{Exh}(\phi)$ *summable-like* if there exists $\epsilon > 0$ such that for each $\delta > 0$ there exists a sequence $(x_n)_n$ of finite subsets of $\mathbb{N}$ with $x_n < x_{n+1}$ and such that for some $k \in \mathbb{N}$ we have

$$\phi(\bigcup_{i \leq k} x_{n_i}) \geq \epsilon \text{ for all } n_0 < \cdots < n_k.$$

Again, one sees easily that this definition does not depend on the choice of ϕ representing I. Of course the classes of density-like and summable-like ideals are disjoint. It is easy to see that the ideal ℓ_1 is summable-like as are the ideals based on Tsirelson's Banach space construction from [2] and [18].

The following simple lemma will not be used in this paper, but it seems appropriate to prove it here.

Lemma 3.2. *Let I be an analytic P-ideal. The following conditions are equivalent*

(i) *I is summable-like;*
(ii) *there exists $\epsilon > 0$ such that for each $\delta > 0$ there is a sequence (x_n) of sets in I with $\phi(x_n) < \delta$ for each n and such that for some $k \in \mathbb{N}$ we have*

$$\phi(\bigcup_{i \leq k} x_{n_i}) \geq \epsilon \textit{ for all } n_0 < \cdots < n_k.$$

Proof. Clearly (i) implies (ii). To see that (ii) implies (i), fix $\epsilon > 0$ as in (ii). We claim that $\epsilon/3$ works for (i). Fix $\delta > 0$. We can assume that

$\delta < \epsilon/3$. Point (ii) allows us to find a sequence (x_n) of elements of I such that $\phi(x_n) < \delta$ and

$$\phi(\bigcup_{i=0}^{k} x_{n_i}) \geq \epsilon$$

for all $n_0 < \cdots < n_k$. As in the proof of Lemma 3.1, we can assume that $x_n \to x$ in $2^{\mathbb{N}}$ and find finite sets $y_n \subseteq x_n \setminus x$ such that $y_n < y_{n+1}$, for each n, and

$$\sum_n \phi\left(x_n \setminus (x \cup y_n)\right) < \frac{\epsilon}{3}.$$

Then for each $n_0 < \cdots < n_k$ we have

$$\begin{aligned}\phi(\bigcup_{i=0}^{k} y_{n_i}) &\geq \phi(\bigcup_{i=0}^{k} x_{n_i}) - \phi(x) - \sum_{i=0}^{k} \phi\left(x_{n_i} \setminus (x \cup y_{n_i})\right)\\ &> \epsilon - \frac{\epsilon}{3} - \frac{\epsilon}{3} = \frac{\epsilon}{3}.\end{aligned}$$

as required. □

3.2. **The Ochan topology.** The topology considered in this subsection was defined in [16, p.3]. A special version of it was developed earlier in [10] as a topology on all closed (or even arbitrary) subsets of a topological space. Even though we will be using only this special version, with an eye to future applications, we chose to present the material and to prove our results for the general notion.

Let D be a topological space with a partial order $\leq$ on it that is closed as a subset of $D \times D$. Let τ stand for the topology on D. The *Ochan topology* on D is the topology whose basis consists of sets of the form

$$[d, U] = \{x \in D \colon d \leq x \text{ and } x \in U\}, \tag{18}$$

where $d \in U \subseteq D$ and U is open with respect to τ. It is easy to see that sets of the above form constitute a basis, that is, that the intersection of two sets as in (18) is a union of such sets. Note that sets X that are dense with respect to the Ochan topology, for short *Ochan dense*, are cofinal everywhere, that is, for each τ-open set U and $d \in U$, there is $d' \in X \cap U$ with $d \leq d'$. Such sets will be particularly important to us; see Corollary 3.4.

The following proposition gives the properties of this topology that will be important for our applications. Recall the definition of a strong Choquet space from [6, 8D]. Given a topological space X the strong Choquet game is played by two players making the following moves: player I plays pairs (U, x) with $x \in U$ and U open in X; player II plays $V \subseteq X$ open. The players take turns, with player I going first, and obey the following rules: if player I played (U, x) in the n-th move, then the n-th move of player II must fulfil $x \in V \subseteq U$; if player II played V in the n-th move, then the $n+1$-th move (U, x) of player I must fulfil $U \subseteq V$. If

$$((U_0, x_0), V_0, (U_1, x_1), V_1, \dots)$$

is a run of the game, player II is declared its winner if $\bigcap_n V_n \neq \emptyset$. The space X is called *strong Choquet* if player II has a winning strategy in the strong Choquet game on X.

Proposition 3.3. *Let D, τ and $\leq$ be as above.*

(i) *The Ochan topology on D refines the original topology τ.*

(ii) *If τ is completely metrizable, then the Ochan topology is strong Choquet.*

Proof. Point (i) is obvious.

(ii) Fix a complete metric ρ on D compatible with τ. We describe a winning strategy σ for II in the strong Choquet game for the Ochan topology. Assume that the moves of the play so far were

$$\begin{aligned} \alpha = (&([d_0, U_0], e_0), [q_0, V_0], \dots, \\ &([d_{n-1}, U_{n-1}], e_{n-1}), [q_{n-1}, V_{n-1}], ([d_n, U_n], e_n)), \end{aligned} \tag{19}$$

where $([d_i, U_i], e_i)$, $i \leq n$, were played by I and $[q_i, V_i]$, $i < n$, by II. We assume that all the moves were legal, that is, that we have

$$d_i \in U_i,\ d_i \leq e_i \in U_i,\ q_i \in V_i,$$
$$d_i \leq q_i \leq e_i,\ e_i \in V_i \subseteq U_i,\ q_i \leq d_{i+1},\ U_{i+1} \subseteq V_i.$$

Additionally we assume that our strategy σ is such that if the move above is played according to it, then $q_i = e_i$, the τ-closure of V_i is included in U_i, and the ρ-diameter of V_i is less than $1/(i+1)$. Now to define σ on position (19), we let $\sigma(\alpha) = [e_n, V_n]$, where V_n is a τ-open set containing e_n, whose τ-closure is contained in U_n, and whose ρ-diameter is less than $1/(n+1)$. This is a legal move of II, in fact, all the conditions mentioned above are maintained.

It is clear that if II uses the above strategy, then, with the notation as in (19), the set $\bigcap_n V_n$ contains precisely one point, call it q, and $q_n \to q$ in τ as $n \to \infty$. Since $q_n \leq q_{n+1}$ for each n and $\leq$ is closed in the product $D \times D$, where D is taken with τ, it follows that $q_n \leq q$ for each n. Thus,

$$q \in \bigcap_n [q_n, V_n]$$

and $\bigcap_n [q_n, V_n]$ is non-empty as required of a winning strategy. □

In the context of the above proposition, it may be worth pointing out that certain other properties of τ, like first countability or regularity, are inherited by the Ochan topology from τ [16, Lemmas 0.0, 0.1].

Corollary 3.4. *Let D, τ and $\leq$ be as above. Assume additionally that τ is Polish. Let Z be a second countable topological space. Let $f\colon D \to Z$ be measurable with respect to the σ-algebra generated by the subsets of D that are analytic with respect to τ. Then there exists $X \subseteq D$ such that X is Ochan dense and $f \restriction X$ is continuous, where X is taken with the Ochan topology.*

Proof. Since in an arbitrary topological space sets having the Baire property form a σ-algebra closed under operation $\mathcal{A}$, see [6, 8.22, 29.14], we get, from Proposition 3.3(i) and the fact that in Polish spaces analytic sets are generated by operation $\mathcal{A}$ applied to closed sets [6, 25.7], that preimages under f of open sets in Z have the Baire property with respect to the Ochan topology. Since, by Proposition 3.3(ii), D with the Ochan topology is Baire, see [6, 8.11, 8.15], and since Z is second countable, the conclusion follows from [6, 8.38]. □

3.3. **Arbitrary Tukey reductions from** NWD **to analytic P-ideals.**

Theorem 3.5. *There is no Tukey function from* NWD *to a density-like ideal.*

We will fix some notation for the remainder of the proof of Theorem 3.5. Let I be a density-like ideal. Let ϕ be a lower semicontinuous submeasure with $I = \mathrm{Exh}(\phi)$. The equivalence of (i) and (ii) of Lemma 3.1 allows us to find, for each $\epsilon > 0$, $\delta(\epsilon) > 0$ such that for each sequence (x_n) of sets in I with $\phi(x_n) < \delta(\epsilon)$ there exist $n_0 < n_1 < \cdots$ such that $\phi(\bigcup_k x_{n_k}) < \epsilon$. We can assume that $\delta(\epsilon) \leq \epsilon$. We can and will assume that $\phi(\mathbb{N}) < 1$ and so d_ϕ−diam$(I) < 1$. Therefore, we can take $\delta(1) = 1$.

We also fix a metric on all compact subsets of $2^{\mathbb{N}}$ that is compatible with the Vietoris topology and is such that that space has diameter less than 1. For compact subsets K, L of $2^{\mathbb{N}}$, we write

$$\mathrm{dist}(K, L)$$

to indicate the value of that metric on the pair (K, L).

We will also consider the Ochan topology on NWD whose basis consists of sets of the form (18). It is easy to see that in the case of NWD, we can actually take the following sets as basic sets for the Ochan topology

$$\{L \in \mathrm{NWD} : K \subseteq L \subseteq U\}, \tag{20}$$

where $K \in \mathrm{NWD}$, $U \subseteq 2^{\mathbb{N}}$ is clopen and $K \subseteq U$. We will denote the set in (20) by $[K, U]$ hoping that the reader can cope with the small notational conflict with (18). For a function g defined only on a subset X of NWD, we will write $g([K, U])$ for $g(X \cap [K, U])$.

The following lemma will form a connection between our results on avoiding families and Tukey functions.

Lemma 3.6. *Let $X \subseteq \mathrm{NWD}$ be Ochan dense, and let $g\colon X \to I$ be continuous, where X is taken with the Ochan topology and I with the submeasure topology. Fix positive real numbers ϵ, δ_j with $j \in \mathbb{N}$. Let $K \in \mathrm{NWD}$ and let $U \subseteq 2^{\mathbb{N}}$ be a non-empty clopen set with $K \subseteq U$ and with $d_\phi{-}\mathrm{diam}(g([K,U])) < \delta(\epsilon)$. Then there exist clopen sets $V_0, \dots, V_k$ and compact sets $L_0, \dots, L_k \in \mathrm{NWD}$ for some $k \in \mathbb{N}$ such that*

(i) $K \subseteq L_j \subseteq V_j$ *for* $j \leq k$*;*
(ii) $\bigcup_{j\leq k} V_j = U$*;*
(iii) $d_\phi{-}\mathrm{diam}(g([L_j, V_j])) < \delta_j$ *for* $j \leq k$*;*
(iv) $\mathrm{dist}(L_j, V_j) < \delta_j$ *for* $j \leq k$*;*
(v) $\phi(\bigcup_{j\leq k} g(L_j) \setminus g(K)) < 2\epsilon$*.*

Proof. It will be convenient to change perspective and talk about covering families rather than avoiding families. By *a covering family* we understand a family of clopen subsets of $2^{\mathbb{N}}$ such that each element of NWD is a subsets of some element of the family. It is clear that the complements of sets in an avoiding family form a covering family and the complements of sets in a covering family form an avoiding family.

For $\epsilon > 0$, define $\mathcal{B}_\epsilon$ to consist of those finite subsets F of I for which

$$(\forall x \in F\ \phi(x) \geq \delta(\epsilon)) \ \text{ or } \ \phi(\bigcup F) < \epsilon.$$

Since $\delta(\epsilon) \leq \epsilon$, $\mathcal{B}_\epsilon$ is a collective on I. The condition from the assumptions of Theorem 2.1 can be rephrased for $\mathcal{B}_\epsilon$ and $Z = I$ as: for each infinite sequence $(x_n)_n$ of elements of I there exist $n_0 < n_1 < \cdots$ such that for each k we have

$$\{x_{n_0}, \dots, x_{n_k}\} \in \mathcal{B}_\epsilon.$$

It is now clear from the definition of $\delta(\epsilon)$ that $\mathcal{B}_\epsilon$ fulfills this condition. Thus, Theorem 2.1 implies that if $\mathcal{F}_n$ are covering families of clopen sets and $f_n\colon \mathcal{F}_n \to I$ with $\phi(f_n(V)) < \delta(\epsilon)$ for all $V \in \mathcal{F}_n$, then there exist $n_0 < \cdots < n_k$ and $V_0 \in \mathcal{F}_{n_0}, \dots, V_k \in \mathcal{F}_{n_k}$ such that

$$\bigcup_{j\leq k} V_j = 2^{\mathbb{N}}$$

and $\{f_{n_1}(V_1), \dots, f_{n_k}(V_k)\} \in \mathcal{B}_\epsilon$. In view of $\phi(f_{n_j}(V_j)) < \delta(\epsilon)$ for each $j \leq k$, this last condition gives

$$\begin{aligned}\phi(\bigcup_{j\leq k} f_{n_j}(V_j)) &\leq \phi(f_{n_0}(V_0)) + \phi(\bigcup_{1\leq j\leq k} f_{n_j}(V_j)) \\ &< \delta(\epsilon) + \epsilon \leq 2\epsilon.\end{aligned} \tag{21}$$

Now let ϵ, δ_j for $j \in \mathbb{N}$, K and U be as in the assumptions. We can assume that $\delta_{j+1} < \delta_j$ for each j and, since U is homeomorphic to $2^{\mathbb{N}}$, that actually $U = 2^{\mathbb{N}}$. We claim that the family $\mathcal{F}_n$ of all clopen sets $V \subseteq U$ for which there exists $L \in$ NWD with

$$K \subseteq L \subseteq V,\ d_\phi\text{−diam}\,(g([L,V])) < \delta_n,\ \text{dist}(L,V) < \delta_n \tag{22}$$

is a covering family. To see this let $L' \subseteq U$ be an arbitrary set in NWD. Since X is Ochan dense, there exists $L \in X$ such that

$$K \cup L' \subseteq L \subseteq U \ \text{ and } \ \text{dist}(L,U) < \delta_n.$$

By continuity of g at L, there exists a clopen set V such that

$$L \subseteq V \subseteq U \ \text{ and } \ d_\phi\text{−diam}\,(g([L,V])) < \delta_n.$$

This V contains L' and is as required by (22).

For $V \in \mathcal{F}_n$, fix $L = L_V$ as in (22). Define

$$f_n(V) = g(L_V) \setminus g(K).$$

Since d_ϕ−diam $(g([K,U])) < \epsilon$, we have

$$\phi(f_n(V)) < \epsilon.$$

Now we apply the statement shown in the first part of this proof to obtain $n_0 < \cdots < n_k$ and $V_0 \in \mathcal{F}_{n_0}, \ldots, V_k \in \mathcal{F}_{n_k}$ such that point (ii) holds and by (21)

$$\phi(\bigcup_{j\le k} f_{n_j}(V_j)) < 2\epsilon. \tag{23}$$

Define $L_j = L_{V_j}$. Then after consulting the definition of f_n, we see that (23) gives point (v). Points (i), (iii), and (iv) follow directly from the choice of L_V and from the inequality $\delta_{n_j} \le \delta_j$ for $j \le k$. □

Proof of Theorem 3.5. Assume there is a Tukey function from NWD to I. By [14, Theorem 5.3(i)], we can assume that the function is measurable with respect to the σ-algebra generated by analytic sets in the Vietoris topology on NWD. Here I is taken with the submeasure topology. Since the Vietoris topology on NWD is Polish, by Corollary 3.4, there exists an Ochan dense set $X \subseteq$ NWD on which our Tukey function is continues if X is taken with the Ochan topology and I is taken with the submeasure topology. Let g stand for the restriction of our Tukey function to the set X.

Fix a double sequence $(\epsilon_j^n)_{j,n\in\mathbb{N}}$ of positive real numbers such that

$$\sum_{j,n} \epsilon_j^n < \infty. \tag{24}$$

Put

$$\delta_j^n = \delta(\epsilon_j^n).$$

By recursion on $n \in \mathbb{N}$, we produce $i_n \in \mathbb{N}$, $K_j^n \in \mathrm{NWD}$ and clopen subsets U_j^n of $2^{\mathbb{N}}$ for $j \leq i_n$ so that

(i) $K_j^n \subseteq U_j^n$ for $j \leq i_n$;

(ii) $\bigcup_{j \leq i_n} U_j^n = 2^{\mathbb{N}}$;

(iii) d_ϕ−diam $\left(g([K_j^n, U_j^n])\right) < \delta_j^n$ for $j \leq i_n$;

(iv) $\mathrm{dist}(K_j^n, U_j^n) < \delta_j^n$ for $j \leq i_n$;

(v) if $n > 0$, then

$$\phi\left(\bigcup_{j \leq i_n} g(K_j^n) \setminus \bigcup_{i \leq i_{n-1}} g(K_i^{n-1})\right) < 2 \cdot \sum_{i \leq i_{n-1}} \epsilon_i^{n-1}.$$

Before proceeding with the construction, we show how to conclude the proof assuming the existence of the objects above. Consider the family

$$\{K_j^n \colon j \leq i_n,\, n \in \mathbb{N}\} \tag{25}$$

of sets in NWD. By properties (ii), (iv), (24) and inequality $\delta_j^n \leq \epsilon_j^n$, the union of the family is dense in $2^{\mathbb{N}}$, so the family is an unbounded subset of NWD. On the other hand,

$$\bigcup_{j \leq i_0} g(K_j^0)$$

is in I as a finite union of sets from that ideal and

$$\left(\bigcup_{n \geq 1} \bigcup_{j \leq i_n} g(K_j^n)\right) \setminus \bigcup_{i \leq i_0} g(K_i^0) = \bigcup_{n \geq 1}\left(\bigcup_{j \leq i_n} g(K_j^n) \setminus \bigcup_{i \leq i_{n-1}} g(K_i^{n-1})\right)$$

is in I by property (v) and by (24). It follows that

$$\bigcup \{g(K_j^n) \colon j \leq i_n,\, n \in \mathbb{N}\} \in I.$$

Thus, the image in I of the family given by (25) is bounded contradicting our assumption that g is the restriction of a Tukey function to X.

We show now how to construct i_n, $K_j^n \in \mathrm{NWD}$ and clopen sets U_j^n for $j \leq i_n$ with properties (i)–(v). We do it by recursion on n. In fact, we will start the construction from $n = -1$ by setting $i_{-1} = 0$, $K_0^{-1} = \emptyset$, $U_0^{-1} = 2^{\mathbb{N}}$, and $\delta_0^{-1} = 1$. Note that (i)–(v) hold ((v) vacuously). Now we explain how to produce these objects for $n \geq 0$ assuming that they have been constructed for smaller values of this parameter. This is done by a recursive construction on $i \leq i_{n-1}$ that produces natural numbers j_i with $j_0 < \cdots < j_i$, closed nowhere dense sets L_j^i and clopen sets V_j^i for $j_{i-1} < j \leq j_i$ with the convention $j_{-1} = -1$. This is done so that

(a) $K_i^{n-1} \subseteq L_j^i \subseteq V_j^i$;

(b) $\bigcup_{j_{i-1} < j \leq j_i} V_j^i = U_i^{n-1}$;

(c) d_ϕ−diam $\Big(g([L^i_j, V^i_j])\Big) < \delta^n_j$;
(d) $\mathrm{dist}(L^i_j, V^i_j) < \delta^n_j$;
(e) $\phi(\bigcup_{j_{i-1}<j\leq j_i} g(L^i_j) \setminus g(K^{n-1}_i)) < 2\epsilon^{n-1}_i$ for $n > 0$.

Such a construction is accomplished by a direct application of Lemma 3.6 with $K = K^{n-1}_i$ and $U = U^{n-1}_i$ and with the values of ϵ and δ_j, $j \in \mathbb{N}$, specified below. It suffices to notice that the assumptions of this lemma are satisfied for these parameters. Indeed, if $n = 0$, this happens with $\epsilon = 1$ and $\delta_j = \delta^0_j$ since

$$d_\phi\text{−diam}\left(g([K^{-1}_0, U^{-1}_0])\right) \leq d_\phi\text{−diam}(I) < 1 = \delta(\epsilon);$$

and if $n > 0$, this is the case with $\epsilon = \epsilon^{n-1}_i$ and $\delta_j = \delta^n_j$ since, by our inductive assumption (iii) on $n-1$, we have

$$d_\phi\text{−diam}\Big(g([K^{n-1}_j, U^{n-1}_j])\Big) < \delta^{n-1}_j = \delta(\epsilon^{n-1}_j).$$

Once the construction above has been performed for all $i \leq i_{n-1}$, we set $i_n = j_{i_{n-1}}$ and, for $j \leq i_n$, $K^n_j = L^i_j$ and $U^n_j = V^i_j$, where i is the unique natural number with $j_{i-1} < j \leq j_i$. Now, for n, points (i), (iii) and (iv) follow from (a), (c) and (d), respectively, point (ii) follows from (b) and the inductive assumption on $n-1$, and point (v) follows from (e) and subadditivity of ϕ. The theorem is proved. □

Using slight modifications of the techniques of Fremlin from [3, Theorem 3B(c), Theorem 2B], we will prove the following theorem. It is somewhat curious that statement so general with respect to I holds true.

Theorem 3.7. *Let I be a summable-like ideal.*

(i) *There is a continuous Tukey reduction from* NWD *to I, where* NWD *is given the Vietoris topology and I the submeasure topology.*
(ii) *Let D be a partial order such that each two elements of D have a maximum. Assume D is equipped with a complete separable metric ρ such that the binary maximum operation is uniformly continuous. Then there is a Borel Tukey function from D to I, where D is taken with the topology given by ρ and I with the submeasure topology.*

Proof. This proof is but a modification of some ideas from [3] to a more general setting. For this reason we will not write down all of its details.

(i) Fix $\epsilon > 0$ such that for each $k \in \mathbb{N}$ we can find a sequence $(x^k_n)_n$ of finite non-empty subsets of $\mathbb{N}$ and $p_k \in \mathbb{N}$ with $x^k_n < x^k_{n+1}$, $\phi(x^k_n) < 2^{-k}$ and such that for each $F \subseteq \mathbb{N}$ with at least p_k elements, we have

$$\phi(\bigcup_{n\in F} x^k_n) > \epsilon.$$

By going to subsequences we can assume that if $(m, l) \neq (n, k)$, then x^l_m and x^k_n are disjoint. Now let $\mathcal{F}_k$ consist of all clopen sets of the form $](s_0, s_1, \ldots, s_{p_k-1})[$, where for each $i < p_k$, s_i is a function from a finite non-empty subset of $\mathbb{N}$ to $\{0, 1\}$, $k \leq \min \mathrm{dom}(s_0)$, and for each $i < p_k - 1$, $\mathrm{dom}(s_i) < \mathrm{dom}(s_{i+1})$. It is not difficult to see that $\mathcal{F}_k$ is an avoiding family and that the intersection of any p_k elements of $\mathcal{F}_k$ is non-empty. Fix a bijective enumeration of $\mathcal{F}_k$:

$$\mathcal{F}_k = \{U^k_n : n \in \mathbb{N}\}.$$

We define a Tukey function $f\colon \mathrm{NWD} \to I$ as follows. For $L \in \mathrm{NWD}$, let $g_L\colon \mathbb{N} \to \mathbb{N}$ be given by letting $g_L(k)$ be the smallest natural number n with

$$U^k_n \cap L = \emptyset.$$

Define

$$f(L) = \bigcup_k x^k_{g_L(k)}.$$

Since $\phi(x^k_{g_L(k)}) < 2^{-k}$, we see that $f(L) \in I$. We leave it to the reader to check that f is a continuous function if NWD is equipped with the Vietoris topology and I with the submeasure topology.

To see that f is Tukey, we need to show that for $a \in I$ the set

$$\{L \in \mathrm{NWD}\colon f(L) \subseteq a\} \tag{26}$$

is bounded in NWD. Since $a \in I$, we have $\phi(a \setminus \{0, \ldots, q\}) \to 0$ as $q \to \infty$ and, therefore, $\phi(a \setminus \{0, \ldots, q\}) < \epsilon$ for large enough q. Thus, there exists k_0 such that for $k \geq k_0$ the set

$$A_k = \{n\colon x^k_n \subseteq a\}$$

has fewer than p_k elements. It follows that the clopen set

$$V^k = \bigcap_{n \in A_k} U^k_n$$

is non-empty. It is also disjoint from each element of family (26). Note also that if $x \in V^k$ and $y \in 2^{\mathbb{N}}$ differs from x only on coordinates less than k, then $y \in V^k$. Therefore, we get that the open set $\bigcup_{k \geq k_0} V^k$ is dense. Since it is disjoint from each set in family (26), the proof of (i) is completed.

(ii) As in point (i) we can fix $\epsilon > 0$, $p_k \in \mathbb{N}$, and finite non-empty subsets x^k_n, $k, n \in \mathbb{N}$, of $\mathbb{N}$ so that if $(m, l) \neq (n, k)$, then x^l_m and x^k_n are disjoint, for each $k \in \mathbb{N}$ we have $\phi(x^k_n) < 2^{-k}$ and for each $F \subseteq \mathbb{N}$ with at least p_k elements

$$\phi(\bigcup_{n \in F} x^k_n) > \epsilon.$$

By uniform continuity with respect to ρ of the maximum operation $\vee$ on D, we can pick $\Delta_k > 0$ such that if $F \subseteq D$ has at most p_k elements,

$F' \subseteq D$ has at most p_{k+1} elements, and for each $d' \in F'$ there is $d \in F$ with $\rho(d, d') < \Delta_k$ and for each $d \in F$ there is $d' \in F'$ with $\rho(d, d') < \Delta_k$, then

$$\rho(\bigvee F, \bigvee F') < 2^{-k}.$$

We will assume that $\Delta_{k+1} < \Delta_k$ and $\Delta_k \to 0$ as $k \to \infty$.

To define a Tukey function from D to I, fix a countable dense subset $R = \{r_n \colon n \in \mathbb{N}\}$ of D. For $d \in D$, let $g_d \colon \mathbb{N} \to \mathbb{N}$ be such that $g_d(k)$ is the first $n \in \mathbb{N}$ with

$$\rho(d, r_n) < \frac{\Delta_k}{2}. \tag{27}$$

This inequality immediately implies

$$\rho(r_{g_d(k)}, r_{g_d(k+1)}) < \Delta_k. \tag{28}$$

Now let $f \colon D \to I$ be given by

$$f(d) = \bigcup_k x^k_{g_d(k)}.$$

We leave it to the reader to check that f is Borel.

To see that f is Tukey, fix $a \in I$. We need to see that

$$\{d \in D \colon f(d) \subseteq a\} \tag{29}$$

is bounded in D. As in point (i), since $a \in I$, there is $k_0 \in \mathbb{N}$ such that for $k \geq k_0$ we have

$$|\{n \in \mathbb{N} \colon x^k_n \subseteq a\}| \leq p_k.$$

It follows that for $k \geq k_0$ we get

$$\begin{aligned} |\{r_{g_d(k)} \colon f(d) \subseteq a\}| &\leq |\{g_d(k) \colon f(d) \subseteq a\}| \\ &\leq |\{n \colon x^k_n \subseteq a\}| \leq p_k. \end{aligned}$$

Thus, by (28) and by the choice of Δ_k, for $k \geq k_0$ we get

$$\rho(\bigvee\{r_{g_d(k)} \colon f(d) \subseteq a\}, \bigvee\{r_{g_d(k+1)} \colon f(d) \subseteq a\}) < 2^{-k}.$$

Since ρ is complete, it follows that the sequence

$$(\bigvee\{r_{g_d(k)} \colon f(d) \subseteq a\})_{k \geq k_0}$$

is convergent to some $d_\infty \in D$. From the definition of d_∞ and definition (27) of g_d, we easily see that $d \leq d_\infty$ for each $d \in D$ with $f(d) \subseteq a$. Thus, d_∞ is the desired bound for set (29). □

Corollary 3.8. (i) *Let I and J be analytic P-ideals. Assume that J is summable-like. Then $I \leq_T J$.*

(ii) *Let I, J be two summable-like ideals. Then $I \equiv_T J$.*

Proof. (i) Since each analytic P-ideal has a complete metric making the union operation uniformly continuous (namely the metric d_ϕ as in (17) coming from a lower semicontinuous submeasure ϕ representing the ideal as $\mathrm{Exh}(\phi)$), point (i) follows immediately from Theorem 3.7(ii).

(ii) follows immediately from (i). □

3.4. **Special types of Tukey reductions from** NWD **to analytic P-ideals.** In the next theorem, we rule out the existence of Tukey reductions mapping bounded sets to bounded sets from NWD to arbitrary analytic P-ideals. Note that the Tukey reduction produced as in Theorem 3.7(i) are continuous, so they map bounded sets to sets in which each sequence has a bounded subsequence (that is, to pseudo-bounded sets). It may also be worth pointing out that $\mathcal{E}_\mu$ does admit a Tukey reduction to $\mathcal{Z}_0$ that is monotone increasing and so maps bounded sets to bounded sets; see the proof of [3, 3K(b)].

Theorem 3.9. *There is no Tukey function from* NWD *to an analytic P-ideal I mapping bounded in* NWD *families of singletons to bounded subsets of I.*

Proof. The conclusion of the theorem will follow from the claim.

Claim. Let X be a metric space, and let I be an analytic P-ideal. Let $f\colon X \to I$ be such that $f(X)$ is unbounded. Then there exists $E \subseteq X$ that is discrete and such that $f(E)$ is unbounded.

To see how the claim implies the theorem, fix a Tukey function $g\colon \mathrm{NWD} \to I$ as in the statement of the theorem, and note that the existence of the function $f\colon 2^{\mathbb{N}} \to I$ given by

$$f(x) = g(\{x\})$$

would contradict the claim.

It suffices to show the claim for a countable metric space X. Assume towards a contradiction that if $E \subseteq X$ is contained in the closure of a discrete set, then $f(E)$ is bounded. Fix also a lower semicontinous submeasure ϕ with $I = \mathrm{Exh}(\phi)$ and a metric d on X. For $r > 0$ and $x \in X$, we write

$$B_r(x) = \{y \in X : d(x,y) < r\}.$$

We first show that for each $x \in X$ we have

$$\forall \epsilon > 0 \exists m \in \mathbb{N} \forall F \subseteq B_{1/m}(x) \text{ finite } (\phi(\bigcup_{y\in F} f(y) \setminus m) < \epsilon). \tag{30}$$

To prove it by contradiction, we assume that the conclusion fails, which allows us to fix $\epsilon > 0$ and pick, for each $m \in \mathbb{N}$, a finite set $F_m \subseteq B_{1/m}(x)$

such that

$$\phi(\bigcup_{y\in F_m} f(y)\setminus m) \geq \epsilon. \tag{31}$$

The set $\bigcup_m F_m$ is the range of a convergent sequence and, therefore, its image $\{f(y) \colon y \in \bigcup_m F_m\}$ is bounded, that is,

$$\bigcup_m \bigcup_{y\in F_m} f(y) \in I.$$

This condition implies that for some $m_0 \in \mathbb{N}$ we have

$$\phi(\bigcup_m \bigcup_{y\in F_m} f(y)\setminus m_0) < \epsilon,$$

contradicting (31) and proving (30).

Fix $k \in \mathbb{N}$. Using (30) and countability of X, we find a clopen disjoint covering

$$\{U^k_n \colon n \in \mathbb{N}\}$$

of X and $m^k_n \in \mathbb{N}$ such that for each $n \in \mathbb{N}$

$$\forall F \subseteq U^k_n \text{ finite } (\phi(\bigcup_{y\in F} f(y)\setminus m^k_n) < 2^{-k-n-2}). \tag{32}$$

For each k, n find a finite set $F^k_n \subseteq U^k_n$ such that

$$\forall y \in U^k_n \; (f(y)\cap m^k_n \subseteq \bigcup_{z\in F^k_n} f(z)). \tag{33}$$

Notice that $\bigcup_n F^k_n$ is discrete since the covering $\{U^k_n : n \in \mathbb{N}\}$ is disjoint. It follows that $f(\bigcup_n F^k_n)$ is bounded in I. Let

$$a_k = \bigcup f(\bigcup_n F^k_n) \in I.$$

From (32) and (33) we conclude that

$$\forall F \subseteq U^k_n \text{ finite } (\phi(\bigcup_{y\in F} f(y)\setminus a_k) < 2^{-k-n-2}).$$

Thus, we have

$$\forall F \subseteq X = \bigcup_n U^k_n \text{ finite } \left(\phi(\bigcup_{y\in F} f(y)\setminus a_k) < \sum_n 2^{-k-n-2} = 2^{-k-1}\right). \tag{34}$$

Let $m_k \in \mathbb{N}$ be such that

$$\phi(a_k \setminus m_k) < 2^{-k-1}.$$

From this inequality and from (34), we get

$$\forall F \subseteq X \text{ finite } (\phi(\bigcup_{y\in F} f(y)\setminus m_k) < 2^{-k-1} + 2^{-k-1} = 2^{-k}. \tag{35}$$

Lower semicontinuity of ϕ and (35) give

$$\phi(\bigcup_{y\in X} f(y)\setminus m_k) < 2^{-k},$$

hence $\bigcup_{y\in X} f(y)$ is in I contradicting our assumption that $f(X)$ is unbounded. □

In [3, 2B], Fremlin proves a general result ensuring Tukey reducibility of a directed order to ℓ_1; in Theorem 3.7(ii) we use his method to do the same for an arbitrary summable-like analytic P-ideal in place of ℓ_1. Fremlin's method is based on the existence of a complete metric making the maximum operation uniformly continuous. We show in the result below that this method cannot be used to construct a Tukey function defined on NWD as long as the metric involved has some connection with the Vietoris topology. Note that $\mathcal{E}_\mu$ does have a metric with the property mentioned above, as do all analytic P-ideals. For analytic P-ideals this is the metric given by (17). For $\mathcal{E}_\mu$ such a metric is defined as follows. For $K\in\mathcal{E}_\mu$, let U^K_n be the smallest under inclusion subset of $2^{\mathbb{N}}$ that is the union of sets of the form $[s]$ with $s\in 2^{<\mathbb{N}}$ and $\mathrm{lh}(s)=n$. (For the definition of $2^{<\mathbb{N}}$ see Subsection 2.1.) For $K,L\in\mathcal{E}_\mu$, set

$$d(K,L)=\sup_n \mu(U^K_n, U^L_n),$$

where μ is the Lebesgue measure on $2^{\mathbb{N}}$. We leave it to the reader to check that d is a complete metric on $\mathcal{E}_\mu$ compatible with the Vietoris topology and making the union operation uniformly continuous.

Theorem 3.10. *There is no complete metric d on* NWD *compatible with the Vietoris topology and making the binary operation of taking union uniformly continuous.*

Proof. Towards a contradiction assume that there exists a metric as in the statement of the theorem. Fix such a metric d on NWD. Using uniform continuity of the union operation, we can find positive real numbers δ^n_i, $n,i\in\mathbb{N}$, so that for all $K_1,K_2\in$ NWD

$$\text{if } d(K_1,K_2)<\delta^n_i,\text{ then } \forall L\in \mathrm{NWD}\ d(L\cup K_1, L\cup K_2)<2^{-n-i-1}.$$

Recursively on n one can construct sequences $s^n_i\in 2^{<\mathbb{N}}$ so that

(i) $\forall n\ \{s^n_i\colon i\in\mathbb{N}\}$ is a maximal antichain in $2^{<\mathbb{N}}$;
(ii) $\forall n,j\exists i\ s^n_i\subsetneq s^{n+1}_j$;
(iii) $d{-}\mathrm{diam}(\{K\in \mathrm{NWD} : \emptyset\neq K\subseteq [s^n_i]\})<\delta^n_i$.

We leave it to the reader to perform this construction. Pick now $x_i^n \in 2^{\mathbb{N}}$ with $s_i^n \subseteq x_i^n$ and define

$$F^n = (2^{\mathbb{N}} \setminus \bigcup_i [s_i^n]) \cup \{x_j^k : k \leq n, j \in \mathbb{N}\}.$$

By (i) and (ii) above, F^n is an element of NWD. Note that F^{n+1} is the increasing union over k of sets $F_k^n \in$ NWD given by

$$F_k^n = F^n \cup \bigcup_{i<k} \left(\bigcup \{([s_i^n] \setminus [s_j^{n+1}]) \cup \{x_j^{n+1}\} \colon s_j^{n+1} \supseteq s_i^n\} \right).$$

Note that $F_{k+1}^n \setminus F_k^n$ is compact, non-empty and is included in $[s_k^n]$ and that $F_k^n \cap [s_k^n]$ is non-empty. It follows by point (iii) and by the choice of δ_k^n, that we have

$$d(F_k^n, F_{k+1}^n) < 2^{-n-k-1}.$$

Therefore, since $F_0^n = F^n$ and $F_k^n \to F^{n+1}$ as $k \to \infty$, we get

$$d(F^{n+1}, F^n) \leq \sum_k 2^{-n-k-1} = 2^{-n},$$

which makes the sequence $(F^n)_n$ d-Cauchy and therefore, by completeness of d, convergent to an element of NWD. Using continuity of union one easily shows that $(F^n)_n$ is bounded. However, it is easy to see using points (i) and (ii) that $\bigcup_n F^n$ is dense in $2^{\mathbb{N}}$, which gives a contradiction. □

A careful reading of the above proof reveals that we use little of the assumption of compatibility of the metric d with the Vietoris topology. The only properties needed for the argument are:

— for each $x \in 2^{\mathbb{N}}$ and each $\epsilon > 0$ there is an open subset U of $2^{\mathbb{N}}$ such that $x \in U$ and the d-diameter of $\{K \in \text{NWD} \colon \emptyset \neq K \subseteq U\}$ is less than ϵ;

— if $F \in$ NWD, (F_k) is an increasing sequence of elements of NWD such that F is the closure of $\bigcup_k F_k$ and (F_k) d-converges, then (F_k) d-converges to F.

3.5. **Unbounded pseudo-bounded sets.** Recall from [5] that a subset A of a directed order is called *pseudo-bounded* if each sequence in A contains a bounded subsequence. When proving Tukey inequivalence of $\mathbb{N}^{\mathbb{N}}$ with $\mathcal{Z}_0$ and with ℓ_1, Isbell [5] used the following property of $\mathcal{Z}_0$ and of ℓ_1: each function from it to itself maps some unbounded set to a pseudo-bounded set. He asked [5, p.653] whether the same property holds for NWD hoping in this way to distinguish $\mathbb{N}^{\mathbb{N}}$ from NWD. The non-reduction NWD $\not\leq_T \mathbb{N}^{\mathbb{N}}$ was later proved in [3]. We use this result to answer Isbell's question in the affirmative. In fact, we prove below a general proposition.

Recall the notion of a basic order, which was introduced in [14]. We call a partial order with a metrizable separable topology *basic* if it has a continuous binary maximum operation, each bounded sequence has a convergent subsequence, and each convergent sequence has a bounded subsequence. Both analytic P-ideals and σ-ideals of compact subsets of a compact metrizable space are basic, see [14]. Recall from [14, Theorem 4.1] that if the topology on a basic order is analytic, then it is Polish.

Proposition 3.11. *Let D be a directed order with $D \not\leq_T \mathbb{N}^{\mathbb{N}}$, and let E be an analytic basic order. Then for each $f\colon D \to E$ there exists an unbounded subset of D whose image under f is pseudo-bounded in E.*

Proof. Let ρ be a complete metric on E compatible with the topology. Fix $n \in \mathbb{N}$ and cover E with sets U_i^n, $i \in \mathbb{N}$, so that

$$\rho\text{−diam}(U_i^n) < \frac{1}{n+1}.$$

Define $g\colon D \to \mathbb{N}^{\mathbb{N}}$ so that for $d \in D$ we have

$$f(d) \in U^n_{g(d)(n)}.$$

Since $D \not\leq_T \mathbb{N}^{\mathbb{N}}$, g is not a Tukey function. It follows that there exists an unbounded set $A \subseteq D$ such that $g(A) \subseteq \mathbb{N}^{\mathbb{N}}$ is bounded by some function in $\mathbb{N}^{\mathbb{N}}$. This condition and completeness of ρ on E easily imply that $f(A)$ is pre-compact. Thus, each sequence of elements of A has a subsequence convergent in E and, therefore, a subsequence bounded in E. It follows that $f(A)$ is pseudo-bounded. □

In connection with the next corollary, recall from [14] that if an analytic basic order is not locally compact (which for analytic P-ideals and analytic σ-ideals of compact sets happens in all but the most trivial cases), then $\mathbb{N}^{\mathbb{N}} \leq_T E$, for example, $\mathbb{N}^{\mathbb{N}} \leq_T$ NWD.

Corollary 3.12. *Let E be an analytic basic order. Assume that $\mathbb{N}^{\mathbb{N}} \leq_T E$. Then $E \not\leq_T \mathbb{N}^{\mathbb{N}}$ if and only if each function from E to E maps some unbounded set to a pseudo-bounded set.*

Proof. The implication $\Rightarrow$ is just Proposition 3.11 applied with $D = E$. The implication $\Leftarrow$ was already essentially known to Isbell. Here is its proof. Suppose that $E \leq_T \mathbb{N}^{\mathbb{N}}$. Recall that $\mathbb{N}^{\mathbb{N}} \leq_T E$ is our background assumption. Fix Tukey functions $f\colon \mathbb{N}^{\mathbb{N}} \to E$ and $g\colon E \to \mathbb{N}^{\mathbb{N}}$. Consider $f \circ g\colon E \to E$. Let $B \subseteq E$ be unbounded. Then $g(B)$ is unbounded in $\mathbb{N}^{\mathbb{N}}$, as g is Tukey, and it is easy to see that there is a coordinate $n \in \mathbb{N}$ such that for some sequence (b_k) in B we have $g(b_k)(n) \to \infty$ as $k \to \infty$. So each subsequence of $(g(b_k))$ is unbounded. Since f is Tukey, each subsequence of $(f \circ g(b_k))$ is unbounded, hence $f \circ g(B)$ is not pseudo-bounded. □

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