Version A

Lecture 13 Solutions

Solutions

1.

A bookstore can obtain a certain gift book from the publisher at a cost of $4 per book. The bookstore has been offering the book at a price of $20 per copy and, at this price, has been selling 150 copies a month. The bookstore is planning to lower its price to stimulate sales and estimates that for each $1 reduction in the price, 15 more books will be sold each month. At what price should the bookstore sell the book to generate the greatest possible profit?

The bookstore should sell the book for $ ____ to maximize their profit.

Solution:

Let x denote the number of $1 reductions in the price and $P(x)$ the corresponding profit function. Then

$$P(x) = (\text{number of books sold}) \times (\text{profit per book})$$

For each $1 reduction in the price, 15 more books than the current 150 will be sold, and so the total number sold will be $150 + 15x$.

Each book will sell for $20 - x$ dollars (the current price minus the number of $1 reductions), and the cost of each book is $4. Hence the profit per book is $(20 - x) - 4 = 16 - x$ dollars. Putting it all together,

$$P(x) = (150 + 15x)(16 - x) = 15(160 + 6x - x^2)$$

The relevant interval is $0 \leq x \leq 16$.

$P\,'(x) = 30(3 - x) = 0$ at $x = 3$.

Now $P(3) = 2535$, $P(0) = 2400$, and $P(16) = 0$, so the greatest profit corresponds to a $3 reduction and the book sells for $17.

Correct answer is: 17

2.

There are 980 yards of fencing available to enclose a rectangular field. How should this fencing be used so that the enclosed area is as large as possible?

The area is maximized at length = ____ yards and width = ____ yards.

Solution:

Let x be the length and y the width of the rectangle.

Then $2x + 2y = 980$ and

$y = 490 - x$.

The area is length × width or

$A(x) = x(490 - x) = 490x - x^2$.

$A'(x) = 490 - 2x = 0$

$A'(x) = 0$ at $x = 245$.

When $x = 245$, and hence $y = 245$.

$A''(x) < 0$.

Hence $A(x)$ is maximized at $x = 245$

Thus the maximum area corresponds to that of a square.

Correct answer is: **The area is maximized at length = 245 yards and width = 245 yards.**

3.

Suppose that x years after its founding in 1992, a certain national consumers' association had a membership of $f(x) = 180(2x^3 - 51x^2 + 420x)$.
At what time between 1992 and 2006 was the membership of the association largest? What was the membership at that time?

The membership was largest with ____ when ____ .

Solution:

The membership of the association x years after 1992 is given by the function
$f(x) = 180(2x^3 - 51x^2 + 420x)$.
The period of time from 1992 to 2006 corresponds to the interval $0 \le x \le 14$.

$$f'(x) = 180(6x^2 - 102x + 420)$$

$$= 1080(x^2 - 17x + 70) = 1080(x - 7)(x - 10).$$

$f'(x) = 0$ when $x = 7$ and $x = 10$.

$f(7) = 202860$, $f(10) = 198000$, and

$f(14) = 246960$.

$f(14) = 246960$ is the absolute maximum. So, the membership was greatest in 2006, fourteen years after the founding of the association, when there were 246960 members.

Correct answer is: The membership was largest with 246960when 2006.

4.

A manufacturer estimates that when q thousand units of a particular commodity are produced each month, the total cost will be $C(q) = 0.4q^2 + 6q + 46$ thousand dollars, and all q units can be sold at a price of $p(q) = 44.4 - 1.2q$ dollars per unit. Determine the level of production that results in maximum profit.

The maximum profit occurs when ____ thousand units are produced.

Solution:

The revenue is

$$R(q) = qp(q) = q(44.4 - 1.2q) = -1.2q^2 + 44.4q$$

thousand dollars, so the profit is

$$P(q) = R(q) - C(q) = -1.2q^2 + 44.4q - (0.4q^2 + 6q + 46)$$

$$= -1.6q^2 + 38.4q - 46$$

thousand dollars. We have

$$P'(q) = -1.6(2q) + 38.4$$

$$= -3.2q + 38.4$$

$$= 0$$

when

$$-3.2q + 38.4 = 0$$

$$q = \frac{38.4}{3.2} = 12$$

Since $P''(q) = -3.2$, it follows that $P''(12) < 0$, and the second derivative test tells us that maximum profit occurs when $q = 12$ (thousand) units are produced.

Correct answer is: 12

5.

A manufacturer estimates that when q thousand units of a particular commodity are produced each month, the total cost will be $C(q) = 0.5q^2 + 7q + 450$ thousand dollars, and all q units can be sold at a price of $p(q) = 22.2 - 1.2q$ dollars per unit.
At what level of production is the average cost per unit

$$A(q) = \frac{C(q)}{q}$$

minimized? What is the minimal average cost?

With production ____ thousand units the average cost is ____ $\frac{\text{dollars}}{\text{unit}}$.

Solution:

The average cost is

$$A(q) = \frac{C(q)}{q} = \frac{0.5q^2 + 7q + 450}{q} \quad \frac{\textit{thousand dollars}}{\textit{thousand units}}$$

$$= 0.5q + 7 + \frac{450}{q} \quad \frac{\textit{dollars}}{\textit{units}}$$

for $q > 0$ (the level of production cannot be negative or zero). You find

$$A'(q) = 0.5 - \frac{450}{q^2} = \frac{0.5q^2 - 450}{q^2}$$

which is 0 for $q > 0$ only when $q = 30$. Since

$$A''(q) = \frac{900}{q^3} > 0 \text{ when } q > 0$$

it follows from the second derivative test for absolute extrema that average cost $A(q)$ is minimized when $q = 30$ (thousand) units. The minimal average cost is

$$A(30) = 0.5(30) + 7 + \frac{450}{30} = 37 \qquad \textit{dollars/unit}$$

Correct answer is: With production 30 thousand units the average cost is 37 $\frac{\text{dollars}}{\text{unit}}$.

6.

A manufacturer estimates that when q thousand units of a particular commodity are produced each month, the total cost will be $C(q) = 0.4q^2 + 24q + 35$ thousand dollars, and all q units can be sold at a price of $p(q) = 177.6 - 1.2q$ dollars per unit. What is the maximum profit?

The maximum profit is $ ____ .

Solution:

The revenue is

$$R(q) = qp(q) = q(177.6 - 1.2q) = -1.2q^2 + 177.6q$$

thousand dollars, so the profit is

$$\begin{aligned} P(q) = R(q) - C(q) &= -1.2q^2 + 177.6q - (0.4q^2 + 24q + 35) \\ &= -1.6q^2 + 153.6q - 35 \end{aligned}$$

thousand dollars. We have

$$\begin{aligned} P'(q) &= -1.6(2q) + 153.6 \\ &= -3.2q + 153.6 \end{aligned}$$

$$= \quad 0$$

when

$$-3.2q + 153.6 = 0$$

$$q = \frac{153.6}{3.2} = 48$$

Since $P''(q) = -3.2$, it follows that $P''(48) < 0$, and the second derivative test tells us that maximum profit occurs when $q = 48$ (thousand) units are produced.

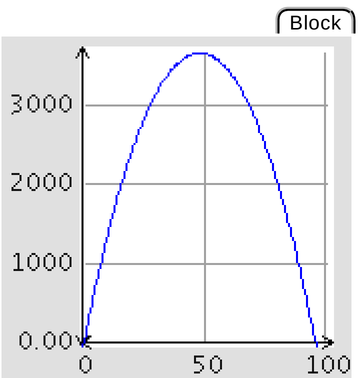
The maximum profit is

$$P(48) = -1.6(48)^2 + 153.6(48) - 35$$

$$= 3651.4$$

thousand dollars ($3651400).

The graph of the profit function is shown below.



The profit function
$P(q) = -1.6q^2 + 153.6q - 35.$

Correct answer is: 3651400

7.

A manufacturer estimates that when q thousand units of a particular commodity are produced each month, the total cost will be $C(q) = 0.6q^2 + 4q + 240$ thousand dollars, and all q units can be sold at a price of $p(q) = 22.2 - 1.2q$ dollars per unit. At what level of production is the average cost equal to the marginal cost $C'(q)$?

The average cost is equal to the marginal cost when q = ____ thousand units.

Solution:

The average cost is

$$A(q) = \frac{C(q)}{q} = \frac{0.6q^2 + 4q + 240}{q} \qquad \frac{\textit{thousand dollars}}{\textit{thousand units}}$$

$$= 0.6q + 4 + \frac{240}{q} \qquad \frac{\textit{dollars}}{\textit{units}}$$

The marginal cost is $C'(q) = 1.2q + 4$, and it equals average cost when

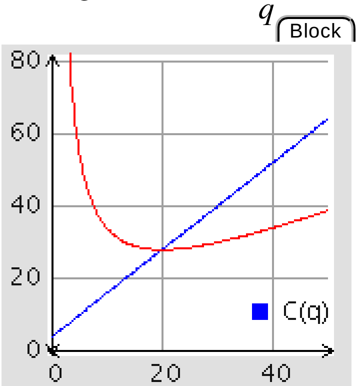
$$1.2q + 4 = 0.6q + 4 + \frac{240}{q}$$

$$0.6q = \frac{240}{q}$$

$$0.6q^2 = 240$$

$$q = 20 \text{ (thousand) units}$$

The graphs of the marginal cost $C'(q)$ and average cost $A = \frac{C(q)}{q}$ are shown below.



Average and marginal cost

Correct answer is: 20

8.

If they exist, find the absolute maximum and absolute minimum of the function

$$f(x) = x^2 + \frac{2}{x} \text{ on the interval } x > 0.$$

Absolute minimum at (____ , ____).

Solution:

The function is continuous on the interval $x > 0$ since its only discontinuity occurs at $x = 0$. The derivative is

$$f'(x) = 2x - \frac{2}{x^2}$$

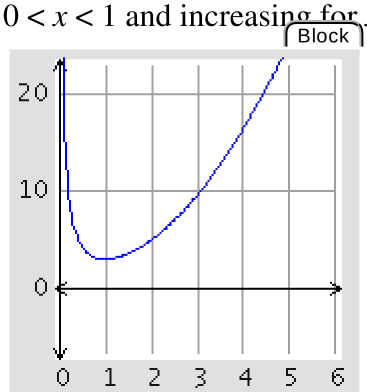
$$= \frac{2x^3 - 2}{x^2}$$

$$= \frac{2(x^3 - 1)}{x^2}$$

which is zero when

$x^3 - 1 = 0 \quad x^3 = 1 \quad \text{or} \quad x = 1$

Since $f'(x) < 0$ for $0 < x < 1$ and $f'(x) > 0$ for $x > 1$, the graph of f is decreasing for $0 < x < 1$ and increasing for $x > 1$, as shown below.



The function $f(x) = x^2 + \frac{2}{x}$ on the interval $x > 0$.

It follows that

$$f(1) = 1^2 + \frac{2}{1} = 3$$

is the absolute minimum of f on the interval $x > 0$ and that there is no absolute maximum.

Correct answer is: Absolute minimum at (1, 3).

