

Lecture 14 Solutions

Solutions

1.

Find two positive numbers whose sum is 4 and whose product is as large as possible.

One is _____ and the other is _____ .

Solution:

Let x be one of the numbers and let y be the other number.

Since their sum is 4, we must have $x + y = 4$.

Their product is $P = xy$, where x and y are both positive. Since

$$x + y = 4 \quad \text{or} \quad y = 4 - x$$

we can eliminate y from the formula for P to obtain a formula in terms of x alone:

$$P(x) = xy = x(4 - x) = 4x - x^2 \quad \text{for } x > 0$$

The derivative of $P(x)$ is

$$P'(x) = 4 - 2x$$

and we get the critical numbers for $F(x)$ by setting $F'(x) = 0$ and solving for x :

$$P'(x) = 4 - 2x = 0$$

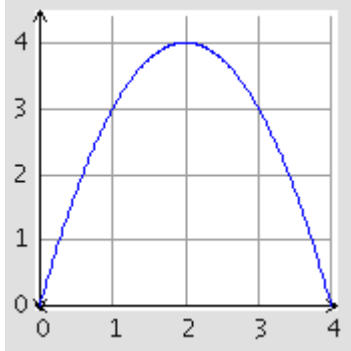
$$2x = 4$$

$$x = 2$$

We can apply the second derivative test for absolute extrema. The second derivative of $F(x)$ is

$$P''(x) = -2$$

so $P''(2) < 0$ and an absolute maximum of $P(x)$ occurs where $x = 2$. For reference, the graph of the function $P(x)$ is shown in the figure below.



The graph of $P(x) = 4x - x^2$ for $x > 0$.

We have shown the maximum is

$$P(2) = 4(2) - (2)^2 = 4$$

which is achieved when one number is 2 and the other number is

$$y = 4 - x = 4 - 2 = 2$$

Correct answer is: One is 2 and the other is 2.

2.

A city recreation department plans to build a rectangular playground having an area of 22500 square meters and surround it by a fence. How can this be done using the least amount of fencing?

- A. The field should have a length of 150 meters and a width of 150 meters.**
- B. The field should have a length of 100 meters and a width of 225 meters.**
- C. The field should have a length of 225 meters and a width of 150 meters.**
- D. The field should have a length of 225 meters and a width of 225 meters.**

Solution:

Let x be the length of the field and y be the width. The amount of fencing is the perimeter of the field, or

$$P = 2x + 2y$$

Since the area is 22500,

$$A = xy$$

$$22500 = xy \text{ or } y = \frac{22500}{x}$$

and

$$P(x) = 2x + 2\left(\frac{22500}{x}\right) = 2x + \frac{45000}{x}$$

which is the function to be minimized.

$$P'(x) = 2 - \frac{45000}{x^2}$$

$$P'(x) = 0 \text{ or } x = 150$$

$$P''(x) = \frac{90000}{x^3}, \text{ so } P''(150) > 0$$

and there is a relative minimum when $x = 150$. Further, since $P''(x) > 0$ for all x in the domain $x > 0$, it is the absolute minimum. So, the field should have a length of 150 meters and a width of 150 meters.

Correct answer is: A

3.

A cylindrical can is to be constructed to hold a fixed volume of liquid. The cost of the material used for the top and bottom of the can is 5 cents per square inch, and the cost of the material used for the curved side is 1 cent per square inch. Use calculus to derive a simple relationship between the radius and height of the can that is the least costly to construct.

A. The height of the can is eleven times its radius.

B. The height of the can is ten times its radius.

C. The height of the can is twice of its radius.

D. The height of the can is equal to its radius.

Solution:

Let r denote the radius, h the height, C the cost (in cents), and V the (fixed) volume. The goal is to minimize cost. We have

$$\text{Cost} = \text{cost of top} + \text{cost of bottom} + \text{cost of side}$$

where, for each component of the cost,

$$\text{Cost} = (\text{cost per square inch})(\text{area})$$

Hence,

$$\text{Cost of top} = \text{cost of bottom} = 5(\pi r^2)$$

and

$$\text{Cost of side} = 1(2\pi rh) = 2\pi rh$$

and the total cost is

$$C = 5\pi r^2 + 5\pi r^2 + 2\pi rh = 10\pi r^2 + 2\pi rh$$

Before you can apply calculus, you must write the cost in terms of just one variable. To do this, use the fact that the can is to have a fixed volume V_0 and solve the equation $V_0 = \pi r^2 h$ for h to get

$$h = \frac{V_0}{\pi r^2}$$

Then substitute this expression for h into the formula for C to express the cost in terms of r alone:

$$C(r) = 10\pi r^2 + 2\pi r \left(\frac{V_0}{\pi r^2} \right) = 10\pi r^2 + \frac{2V_0}{r}$$

The radius r can be any positive number, so the goal is to find the absolute minimum of $C(r)$ for $r > 0$. Differentiating $C(r)$, you get

$$C'(r) = 20\pi r - \frac{2V_0}{r^2} \quad \text{remember } V_0 \text{ is a constant}$$

Since $C'(r)$ exists for all $r > 0$, any critical number $r = R$ must satisfy $C'(R) = 0$; that is,

$$C'(R) = 20\pi R - \frac{2V_0}{R^2} = 0$$

$$\begin{aligned}
 R^2 & \\
 20\pi R &= \frac{2V_0}{R^2} \\
 R^3 &= \frac{2V_0}{20\pi} \\
 R &= \sqrt[3]{\frac{V_0}{10\pi}}
 \end{aligned}$$

If H is the height of the can that corresponds to the radius R , then $V_0 = \pi r^2 H$, and since R must satisfy

$$20\pi R = \frac{2V_0}{R^2}$$

you find that

$$20\pi R = \frac{2(\pi R^2 H)}{R^2} = 2\pi H$$

or

$$H = \frac{20\pi R}{2\pi} = \frac{V_0}{\pi R^2} = 10R$$

Finally, note that the second derivative of $C(R)$ satisfies

$$C''(r) = 20\pi + \frac{4V_0}{r^3} > 0 \quad \text{for all } r > 0$$

Therefore, since $r = R$ is the only critical number for $C(r)$ and since $C''(R) > 0$, the second derivative test for absolute extrema assures you that when the total cost of constructing the can is minimized, the height of the can is ten times its radius.

Correct answer is: B

4.

A carpenter has been asked to build an open box with a square base. The sides of

the box will cost \$12 per square meter, and the base will cost \$16 per square meter. What are the dimensions of the box of greatest volume that can be constructed for \$768?

_____ meters by _____ meters by _____ meters

Solution:

Let x be the length of the side of the square base and y be the height of the box. The volume of the box is

$$V = x^2y$$

The cost of the four sides is

$$4(\text{cost per unit area})(\text{area}) = 4(12)(xy) = 48xy$$

The cost of the base is

$$(\text{cost per unit area})(\text{area}) = 16(x^2)$$

Since there is 768 dollars available to build the box,

$$768 = 48xy + 16x^2$$

or

$$y = \frac{768 - 16x^2}{48x} = \frac{16}{x} - \frac{x}{3}$$

and

$$V(x) = x^2 \left(\frac{16}{x} - \frac{x}{3} \right) = 16x - \frac{1}{3}x^3$$

which is the function to be maximized. The derivative of $V(x)$ is

$$V'(x) = 16 - x^2$$

and we get the critical numbers for $V(x)$ by setting $V'(x) = 0$ and solving for x :

$$V'(x) = 16 - x^2 = 0$$

$$x^2 = 16$$

$$x = 4 \quad \text{rejecting } -4 \text{ since } x > 0$$

Since $x = 4$ is the only critical number in the interval $x > 0$, we can apply the second derivative test for absolute extrema. The second derivative of $F(x)$ is

$$V''(x) = -2x$$

so $V''(x) < 0$ and there is a relative maximum when $x = 4$. Further, since $V''(x) < 0$ for all x in

the domain $x > 0$, it is the absolute maximum. So the box has a maximum volume when its dimensions are 4 meters by 4 meters by

$$y = \frac{16}{4} - \frac{4}{3} = \frac{8}{3} \text{ meters}$$

Correct answer is: 4 meters by 4 meters by $\frac{8}{3}$ meters

5.

A cable is to be run from a power plant on one side of a river 2400 meters wide to a factory on the other side, 1900 meters downstream. The cost of running the cable under the water is \$50 per meter, while the cost over land is \$40 per meter. What is the most economical route over which to run the cable? Round your answer to the nearest integer.

The most economical route way is to run a cable under the water ____ meters and ____ meters over the land.

Solution:

The cable is first run under water to a point at a distance x meters from that point directly across the river from the power plant. The cable is then run over land the remaining $1900 - x$ meters to the factory.

We need to minimize the cost

$C(x) = \text{cost under water} + \text{cost over land.}$

Since the distance over land is $1900 - x$ and

the distance under water is

$$\sqrt{x^2 + (2400)^2} \quad (\text{by the Pythagorean theorem}),$$

the total cost is

$$C(x) = 50\sqrt{x^2 + (2400)^2} + 40(1900 - x).$$

Thus, $C'(x) = \frac{50x}{\sqrt{x^2 + (2400)^2}} - 40$

when $50x = 40\sqrt{x^2 + (2400)^2}$

$$25x^2 = 16[x^2 + (2400)^2]$$

$$x^2 = \frac{16(2400)^2}{9} \text{ or } x = \pm 3200$$

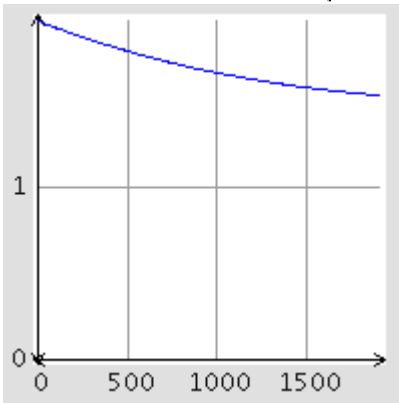
neither of which is in $0 \leq x \leq 1900$.

The absolute minimum occurs at an endpoint.

Now $C(0) = 50(2400) + 40(1900) = 196000$

$$C(1900) = 50\sqrt{(1900)^2 + (2400)^2} \approx 153053$$

The minimum cost of $\$153053$ occurs when the cable is run entirely under water.



Correct answer **The most economical route way is to run a cable under the water 1900 meters and 0 meters over the land.**

6.

What is the maximum possible volume of a cylindrical can with no top that can be made from 243π square inches of metal?

_____ cubic inches

Solution:

The volume of a can is

$$V = \pi r^2 h$$

The amount of metal for the can is 243π , so

$$m = \pi r^2 + 2\pi r h$$

$$243\pi = \pi r^2 + 2\pi r h$$

or

$$h = \frac{243 - r^2}{2r}$$

and

$$V(r) = \pi r^2 \left(\frac{243 - r^2}{2r} \right) = \frac{243\pi}{2} r - \frac{\pi}{2} r^3$$

which is the function to be maximized. The derivative of $V(r)$ is

$$V'(r) = \frac{243\pi}{2} - \frac{3\pi}{2} r^2$$

and we get the critical numbers for $V(r)$ by setting $V'(r) = 0$ and solving for r :

$$V'(r) = \frac{243\pi}{2} - \frac{3\pi}{2} r^2 = 0$$

$$\frac{3\pi}{2} r^2 = \frac{243\pi}{2}$$

$$r^2 = 81$$

$$r = 9 \quad \text{reject } -9 \text{ since } r > 0$$

Since $r = 9$ is the only critical number in the interval $r > 0$, we can apply the second derivative test for absolute extrema. The second derivative of $V(r)$ is

$$V''(r) = -3\pi r$$

So $V''(9) < 0$ and there is a relative maximum when $r = 9$. Further, since $V''(r) < 0$ for all r in the domain $r > 0$, it is an absolute maximum. So, when the can's radius is 9 inches, it has its maximum volume of

$$\frac{243\pi}{2} (9) - \frac{\pi}{2} (9)^3 = 729\pi \text{ cubic inches}$$

Correct answer is: 729π cubic inches

7.

Use the fact that 240 fluid ounces is approximately 137.8π cubic inches to find the dimensions of the 12-ounce soda can that can be constructed using the least amount of metal. What are the dimensions of the least amount of metal? Round your answer to two decimal places.

Block

Radius: ____ inches, Height of can: ____ inches

Solution:

The amount of material is the amount for the circular top and bottom, and the amount for the curved side.

$$m = 2\pi r^2 + 2\pi rh$$

Because 240 fluid ounces is approximately 137.8π cubic inches, 12-ounce is approximately 6.89π cubic inches,

$$V = \pi r^2 h$$

$$6.89\pi = \pi r^2 h, \text{ or } h = \frac{6.89}{r^2}$$

$$\text{and } m(r) = 2\pi r^2 + 2\pi r \left(\frac{6.89}{r^2} \right)$$

$$= 2\pi r^2 + \frac{13.78\pi}{r}$$

which is the function to be minimized.

$$m'(r) = 4\pi r - \frac{13.78\pi}{r^2}$$

$$m'(r) = 0 \text{ when } r \approx 1.51$$

$$m''(r) = 4\pi + \frac{27.56}{r^3},$$

$$\text{so } m''(1.51) > 0$$

and there is a relative minimum when $r = 1.51$.

Further, since $m''(r) > 0$ for all r in the domain $r > 0$, it is an absolute minimum.

So, the minimum material is when the can's radius is approximately 1.51 inches

and its height is approximately $\frac{6.89}{(1.51)^2} \approx 3.02$ inches.

(These dimensions are not used due to packing and handling concerns.)

Correct answer is: Radius:1.51inches, Height of can:3.02inches
