Version A

lecture 17

Solutions

1.

Differentiate the given function. Simplify your answer.

$$f(x) = \frac{\ln x}{2x}$$

$f(x) = \frac{\_\_\_\_}{\_\_\_\_}$

Solution:

$$f'(x) = \frac{2x \frac{d}{dx}(\ln x) - (\ln x)\frac{d}{dx}2x}{4x^2}$$

$$= \frac{1 - \ln x}{2x^2}$$

Correct answer is: $f(x) = \frac{-\ln x + 1}{2x^2}$

2.

Differentiate the given function.

$$f(x) = 62e^x \ln x$$

$f'(x) =$ ____

Solution:

$$f'(x) = 62e^x\left(\frac{1}{x}\right) + 62e^x \ln x$$

$$= 62e^x\left(\frac{1}{x} + \ln x\right)$$

Correct answer is: $f'(x) = 62e^x(x^{-1} + \ln x)$

3.

Find an equation for the tangent line to $m = f(x)$ at the specified point.

$f(x) = 14x^2 \ln \sqrt{x}$; where $x = 1$

$y =$ ____

Solution:

$f(x) = 14x^2 \ln x^{1/2} = 7x^2 \ln x$

$$f'(x) = \left(7x^2\right)\left(\frac{1}{x}\right) + (14 \ln x)(x)$$

$$= 7x + 14x \ln x$$

So, $m = f'(1) = 7 + 14 \ln 1 = 7$.

Also, $f(1) = 14 \ln 1 = 0$, so the point $(1, 0)$ is on tangent line and

$y - 0 = 7(x - 1)$,

or $y = 7x - 7$.

Correct answer is: $y = 7x - 7$

4.

Find the second derivative of the given function.

$f(x) = e^{2x} + 28e^{-x}$

$f''(x) =$ ____

Solution:

$$f'(x) = e^{2x} \bullet (2) + 28e^{-x} \bullet (-1)$$
$$= 2e^{2x} - 28e^{-x}$$
$$f''(x) = 2e^{2x} \bullet (2) - 28e^{-x} \bullet (-1)$$
$$= 4e^{2x} + 28e^{-x}$$

Correct answer is: $f''(x) = 4e^{2x} + 28e^{-x}$

5.

Find the second derivative of the given function.

$f(x) = \ln(20x) + 8x^2$

$f''(x) =$ ____

Solution:

$$f'(x) = \frac{20}{20x} + 16x$$
$$= \frac{1}{x} + 16x$$
$$f''(x) = -\frac{1}{x^2} + 16$$

Correct answer is: $f''(x) = \frac{-1}{x^2} + 16$

6.

Differentiate the function:

$$y = 15^x$$

$$\frac{dy}{dx} = \_\_\_\_$$

Solution:

To differentiate $y = 15^x$, we use logarithmic differentiation as follows:

$y = 15^x$		
$\ln y = x\,(\ln 15)$		*take natural logarithms on both sides*
$\frac{1}{y}\frac{dy}{dx} = \ln 15$		*differentiate both sides*
$\frac{dy}{dx} = (\ln 15)\, y$		*multiply both sides by y*
$\frac{dy}{dx} = (\ln 15)\, 15^x$		*substitute* $y = 15^x$

Correct answer is: $\frac{dy}{dx} = (\ln 15)\, 15^x$

7.

Differentiate the function.

$$y = \log_{12} x$$

$$\frac{dy}{dx} = \_\_\_\_$$

Solution:

We find

$y = \log_{12} x$		
$12^y = x$	*definition of logarithm*
$y \ln 12 = \ln x$	*take natural logarithms on both sides*
$(\ln 12)\dfrac{dy}{dx} = \dfrac{1}{x}$	*differentiate both sides*
$\dfrac{dy}{dx} = \dfrac{1}{x(\ln 12)}$	*divide by* ln 12

Correct answer is: $\dfrac{dy}{dx} = \dfrac{1}{x(\ln 12)}$

8.

The cost $C(x)$ of producing x units of a particular commodity is given. Find the marginal cost $C'(x)$.

$\mathbf{C(x) = 4500e^{0.08x}}$

$C'(x) =$ ____

Solution:

$$C'(x) = 4500(0.08)e^{0.08x}$$
$$= 360e^{0.08x}$$

Correct answer is: $C'(x) = 360e^{0.08x}$

9.

Differentiate the function $f(x) = \sqrt{x+4}$.

$$(2 - 8x)^7$$

A. $f'(x) = \frac{1}{8}\frac{1}{x + 4} + \frac{56}{2 - 8x}$

B. $f'(x) = \left[\frac{\sqrt{x + 4}}{(2 - 8x)^7}\right]\left[\frac{1}{8}\frac{1}{x + 4} - \frac{56}{2 - 8x}\right]$

C. $f'(x) = \left[\frac{\sqrt{x + 4}}{(2 - 8x)^7}\right]\left[\frac{1}{8}\frac{1}{x + 4} + \frac{56}{2 - 8x}\right]$

D. $f'(x) = \left[\frac{x + 4}{(2 - 8x)^7}\right]\left[\frac{1}{8}\frac{1}{x + 4} + \frac{56}{2 - 8x}\right]$

Solution:

You could do this problem using the quotient rule and the chain rule, but the resulting computation would be somewhat tedious.

A more efficient approach is to take logarithms of both sides of the expression for f:

$$\ln f(x) = \ln\left[\frac{\sqrt{x + 4}}{(2 - 8x)^7}\right] = \ln \sqrt{x + 4} - \ln (2 - 8x)^7$$

$$= \frac{1}{2} \ln (x + 4) - 7 \ln (2 - 8x)$$

(Notice that by introducing the logarithm, you eliminate the quotient, the square root, and the seventh power.)

Now use the chain rule for logarithms to differentiate both sides of this equation to get

$$\frac{f'(x)}{f(x)} = \frac{1}{2}\frac{1}{(x + 4)} - 7\left(\frac{-8}{2 - 8x}\right) = \frac{1}{2}\frac{1}{x + 4} + \frac{56}{2 - 8x}$$

so that

$$f'(x) = f(x)\left[\frac{1}{2}\frac{1}{x+4} + \frac{56}{2-8x}\right]$$

$$= \left[\frac{\sqrt{x+4}}{(2-8x)^7}\right]\left[\frac{1}{2}\frac{1}{x+4} + \frac{56}{2-8x}\right]$$

Correct answer is: C

10.

A demographer studying a certain community uses an exponential model $P(t) = P_0e^{kt}$ for the population, where t is the number of years after 1990. If the population is 100000 in 1990 and 138000 in 1998, at what annual percentage rate is the population increasing? Round your final answer to two decimal places.

The population is increasing at the annual rate of ____ %.

Solution:

For simplicity, measure the population $P(t)$ in thousands of individuals. Since the population is 100000 when $t = 0$ (1990) and 138000 when $t = 8$ (1998), we have

$$P_0 = P(0) = 100$$

and

$$P(8) = 100e^{k(8)} = 138$$

$$e^{k(8)} = \frac{138}{100} = 1.38$$

$$\ln\left(e^{k(8)}\right) = \ln(1.38) \qquad \textit{take logarithms on both sides}$$

$$k(8) = 0.3221 \qquad \textit{use } \ln(e^x) = x$$

$$k = \frac{0.3221}{8} = 0.0403$$

Thus, the population is increasing at the annual rate of $100k = 4.03\%$.

Correct answer is: 4.03

11.

The following data were compiled by a researcher during the first 15 minutes of an experiment designed to study the growth of bacteria:

Number of minutes	**0**	**15**
Number of bacteria	**16000**	**20000**

Assuming that the number of bacteria grows exponentially, how many bacteria will be present after 45 minutes?

Your Answer: ____ bacteria will be present after 45 minutes.

Solution:

Since the growth is exponential, $P(t) = P_0 e^{kt}$ where the initial number of bacteria is $P_0 = 16000$.

Also, $P(15) = 16000e^{15k} = 20000$, so $e^{15k} = \frac{5}{4}$.

$$\begin{aligned} \text{Now, } P(45) &= 16000(e^{45k}) \\ &= 16000(e^{15k})^3 \\ &= 16000\left(\frac{5}{4}\right)^3 \\ &= 31250 \text{ bacteria} \end{aligned}$$

Correct answer is: 31250

12.

The amount of a sample of a radioactive substance remaining after *t* years is given by a function of the form $Q(t) = Q_0 e^{-0.0001t}$. At the end of 5,000 years, 287 grams of the substance remain. How many grams were present initially? Round

your answer to the nearest hundredth gram.

Your Answer: ____ grams

Solution:

$Q(t) = Q_0 e^{-0.0001t}$

When $t = 0$, $Q = Q_0$ grams and when $t = 5000$ years $Q = 287$ grams.

$287 = Q_0 e^{(-0.0001)(5000)} = Q_0 e^{-.5}$ or $Q_0 = 287e^{.5} = 473.18$ grams.

Correct answer is: 473.18

13.

A radioactive substance decays exponentially. If 600 grams of the substance were present initially and 200 grams are present 50 years later, how many grams will be present after 150 years? Round your answer to the nearest tenth of a gram if needed.

Your Answer: ____ grams

Solution:

Since the decay is exponential and 600 grams were present initially,

$$Q(t) = 600e^{-kt}$$

Also $Q(50) = 600e^{-50k} = 200$, so $e^{-50k} = \frac{1}{3}$

$$\begin{aligned} \text{Now, } Q(150) &= 600e^{-150k} \\ &= 600\left(e^{-50k}\right)^3 \\ &= 600\left(\frac{1}{3}\right)^3 \\ &= 22.2 \text{ grams} \end{aligned}$$

Correct answer is: 22.2