Version A

Lecture 22 Solutions

Solutions

1.

Find the area of the region *R* enclosed by the curves $y = x^2$ and $y = 4x$.

The area of *R* is ____ .

Solution:

To find the points where the curves intersect, solve the equations simultaneously as follows:

$$x^2 = 4x$$

$$x^2 - 4x = 0 \qquad \textit{subtract } 4x \textit{ from both sides}$$

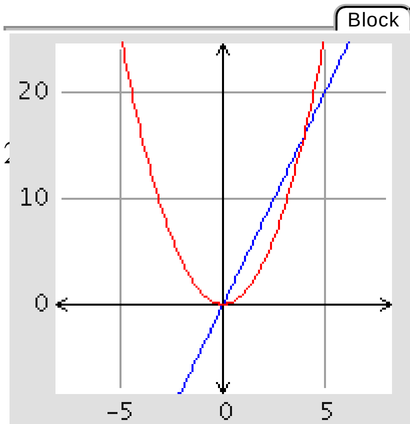
$$x(x-4) = 0 \qquad \textit{factor out } x$$

$$x = 0, 4 \qquad uv = 0 \textit{ if and only if } u = 0 \textit{ or } v = 0$$

The corresponding points (0, 0) and (4, 16) are the only points of intersection.

The region *R* enclosed by the two curves is bounded above by $y = 4x$ and below by $y = x^2$ over the interval $0 \le x \le 4$ (see the figure below).

Correct answer is: The area of *R* is $\frac{32}{3}$.



2.

...region *R*.

...curves $y = x^3 - x^2$ and $y = x^2 + 15x$.

Area = ____

The area of this region is given by the integral

Solution:

The points of intersection are

$$x^3 - x^2 = x^2 + 15x$$

$$x^3 - 2x^2 - 15x = 0$$

$$x(x - 5)(x + 3) = 0.$$

There are two shaded areas

$$\int_{-3}^{0} \left[x^3 - 2x^2 - 15x\right] dx + \int_{0}^{5} \left[x^3 - 2x^2 - 15x\right] dx$$

$$= \left(\frac{x^4}{4} - \frac{2x^3}{3} - \frac{15x^2}{2} \right) \Big|_{-3}^{0} + \left(\frac{x^4}{4} - \frac{2x^3}{3} - \frac{15x^2}{2} \right) \Big|_{0}^{5}$$

$$= \frac{117}{4} + \frac{1375}{12} = \frac{863}{6}$$

Correct answer is: Area = $\frac{863}{6}$

3.

Sketch the given region R and then find its area.

R is the region bounded by the curve $y = 4x^2 - 12x$ and the x−axis. [Hint: Note that the region is below the x−axis.]

Area = ____

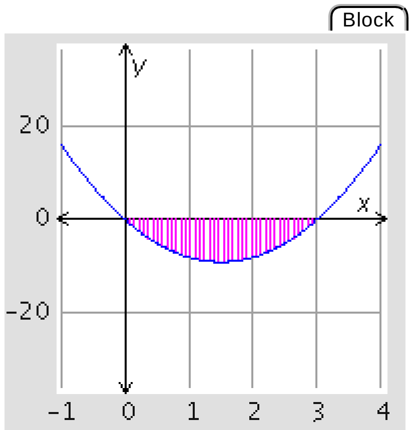
Solution:

The shaded area is

$$\int_{0}^{3} \left[0 - (4x^2 - 12x)\right] dx$$

$$= -\frac{4}{3}x^3 + 6x^2 \Big|_{0}^{3}$$

$$= 18$$



Correct answer is: Area = $\frac{54}{3}$

4.

The demand and supply functions, $D(q)$ and $S(q)$, for a particular commodity are given. Specifically, q thousand units of the commodity will be demanded (sold) at a price of $p = D(q)$ dollars per unit, while q thousand units will be supplied by producers when the price is $p = S(q)$ dollars per unit. Find the equilibrium price p_e (where supply equals demand).

$$D(q) = 229 - \frac{1}{4}q^2;\ S(q) = 85 + \frac{3}{4}q^2$$

p_e = $ ____

Solution:

The supply equals demand when

$$229 - \frac{1}{4}q^2 = 85 + \frac{3}{4}q^2$$

$$q^2 = 144 \text{ or } q = 12$$

So, the equilibrium price is

$$p_e = D(12) = 229 - \frac{1}{4}(12)^2 = \$193$$

Correct answer is: 193

5.

$p = D(q)$ is the price (dollars per unit) at which q units of a particular commodity will be demanded by the market (that is, all q units will be sold at this price), and q_0 is a specified level of production. Find the price $p_0 = D(q_0)$ at which q_0 units will be demanded so that you can compute the corresponding consumers' surplus CS.

$$D(q) = 4(51 - q^2);\ q_0 = 3 \text{ units}$$

The consumers' surplus is $ ____ .

Solution:

$$\begin{aligned} p_0 &= D(q_0) \\ &= 4[51 - (3)^2] \\ &= 168 \text{ dollars per unit} \end{aligned}$$

Using $p = 168$ and $q_0 = 3$, we find that the consumer's surplus is

$$\begin{aligned} CS &= \int_0^3 [4(51 - q^2)]\, dq - (168)(3) \\ &= \left[4\left(51q - \frac{q^3}{3}\right)\right]\Bigg|_0^3 - (168)(3) \\ &= 576 - 504 = 72 \end{aligned}$$

Thus, the consumers' surplus is $72.

Correct answer is: 72

6.

Find the volume of the solid of revolution formed by rotating the region R about

the x−axis.

***R* is the region under the curve $y = x^2 + 2$ from $x = -3$ to $x = 2$.**

The volume is ____ π

Solution:

$$\text{Volume of } S = \pi \int_{-3}^{2} (x^2 + 2)^2 \, dx$$

$$= \pi \int_{-3}^{2} x^4 + 4x^2 + 4 \, dx$$

$$= \pi \left(\frac{x^5}{5} + \frac{4x^3}{3} + 4x \right) \Big|_{-3}^{2}$$

$$= \pi \left[\left(\frac{32}{5} + \frac{32}{3} + 8 \right) - \left(-\frac{243}{5} - \frac{108}{3} - 12 \right) \right]$$

$$= \pi \left[\frac{96}{15} + \frac{160}{15} + \frac{120}{15} + \frac{729}{15} + \frac{540}{15} + \frac{180}{15} \right]$$

$$= \frac{365}{3} \pi$$

Correct answer is: The volume is $\frac{1825}{15} \pi$

7.

Find the volume of the solid of revolution formed by rotating the region *R* about the x−axis.

***R* is the region under the line $y = e^{-0.7x}$ from $x = 0$ to $x = 14$. Round your answer to the third decimal place.**

The volume is approximately ____ .

Solution:

$$\text{Volume} = \pi \int_0^{14} (e^{-0.7x})^2\, dx$$

$$= \pi \int_0^{14} e^{-1.4x}\, dx$$

$$= \frac{\pi}{-1.4} e^{-1.4x} \Big|_0^{14}$$

$$= -\frac{\pi}{1.4}\left(e^{-19.6} - 1\right)$$

$$\approx 2.244$$

Correct answer is: 224

8.

Find the volume of the solid of revolution formed by rotating the region *R* about the *x*−axis.

***R* is the region under the line $y = 6x + 9$ from $x = 0$ to $x = 2$.**

The volume is ____ π.

Solution:

$$\text{Volume of } S = \pi \int_0^{2} (6x + 9)^2\, dx$$

$$= \pi \int_0^{2} 36x^2 + 108x + 81\, dx$$

$$= \pi\left(12x^3 + 54x^2 + 81x\right) \Big|_0^{2}$$

$$= \pi\left[(96 + 216 + 162) - (0)\right] = 474\pi$$

Correct answer is: 474

9.

The demand and supply functions, $D(q)$ and $S(q)$, for a particular commodity are given. Specifically, q thousand units of the commodity will be demanded (sold) at a price of $p = D(q)$ dollars per unit, while q thousand units will be supplied by producers when the price is $p = S(q)$ dollars per unit. Find the consumers' surplus and the producers' surplus at equilibrium.

$D(q) = 135 - \frac{1}{3}q^2$; $S(q) = 54 + \frac{2}{3}q^2$

***CS* = $ ____**

***PS* = $ ____**

Solution:

The supply equals demand when

$$135 - \frac{1}{3}q^2 = 54 + \frac{2}{3}q^2$$

$$q^2 = 81 \text{ or } q = 9$$

So, the equilibrium price is

$$p_e = D(9) = 135 - \frac{1}{3}(9)^2 = \$108$$

The corresponding consumer's surplus is

$$CS = \int_0^9 135 - \frac{1}{3}q^2\,dq - 9(108)$$

$$= \left(135q - \frac{1}{9}q^3\right)\Big|_0^9 - 972$$

$$= \$162$$

and the corresponding producer's surplus is

$$PS = (9)(108) - \int_0^9 54 + \frac{2}{3}q^2\,dq$$

$$= 972 - \left(54q + \frac{2}{9}q^3\right)\Big|_0^9$$

$$= \$324$$

Correct **The demand and supply functions, $D(q)$ and $S(q)$, for a particular commodity are given.**

Specifically, q thousand units of the commodity will be demanded (sold) at a price of $p = D(q)$ dollars per unit, while q thousand units will be supplied by producers when the price is $p = S(q)$ dollars per unit. Find the consumers' surplus and the producers' surplus at equilibrium.

answer is:

$$D(q) = 135 - \frac{1}{3}q^2;\ S(q) = 54 + \frac{2}{3}q^2$$

$CS = \$162$

$PS = \$324$

10.

A manufacturer estimates that q (thousand) products will be purchased (demanded) by wholesalers when the price is

$$p = D(q) = -0.1q^2 + 1080$$

dollars per unit, and the same number of products will be supplied when the price is

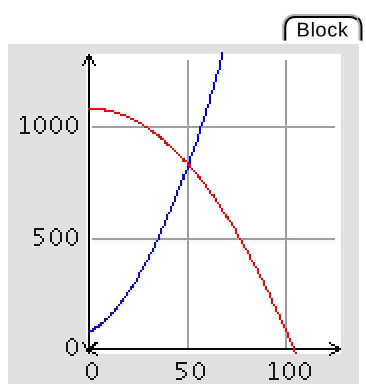
$$p = S(q) = 0.2q^2 + 5q + 80$$

dollars per unit. Determine the consumers' surplus at the equilibrium price. Round your intermediate calculations to two decimal places.

The consumers' surplus is \$ ____ .

Solution:

The supply and demand curves are shown to the right.



Supply equals demand when

$$-0.1q^2 + 1080 = 0.2q^2 + 5q + 80$$

$$0.3q^2 + 5q - 1000 = 0$$

$$q = 50 \quad (\text{reject } q \approx -66.67)$$

and $p = -0.1(50)^2 + 1080 = 830$ dollars per unit. Thus, equilibrium occurs at a price of \$830 per unit, and then 50000 products are supplied and demanded.

Using $p_0 = 830$ and $q_0 = 50$, we find that the consumers' surplus is

$$\text{CS} = \int_0^{50} (-0.1q^2 + 1080)\, dq - (830)(50)$$

$$= \left[-0.1\left(\frac{q^3}{3}\right) + 1080q\right]\Bigg|_0^{50} - (830)(50)$$

$$\approx 49833.33 - 41500 = 8333.33$$

or \$8333330 (since $q_0 = 50$ is really 50000).

Correct answer is: 8333330
