**Version A**

# lecture 23 Practice

## Solutions

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1.

**Money is transferred continuously into an account at the constant rate of $1500 per year. The account earns interest at the annual rate of 5% compounded continuously. How much will be in the account at the end of 4 years? Round your final answer to the nearest cent.**

**The future value of the income stream is $ ____ .**

Solution:

Note that $P$ dollars invested at 5% compounded continuously will be worth $Pe^{0.05t}$ dollars $t$ years later.

To approximate the future value of the income stream, divide the 4−year time interval $0 \le t \le 4$ into $n$ equal subintervals of length $\Delta t$ years and let $t_k$ denote the beginning of the $k$th subinterval. Then, during the $k$th subinterval (of length $\Delta t$ years),

$$\text{Money deposited} = (\text{dollars per year})(\text{number of years}) = 1500\Delta t$$

If all of this money were deposited at the beginning of the subinterval (at time $t_k$), it would remain in the account for $4 - t_k$ years and therefore would grow to $(1500\Delta t)e^{0.05(4 - t_k)}$ dollars. Thus,

$$\begin{array}{c}\text{Future value of}\\ \text{money deposited}\\ \text{during } k\text{th subinterval}\end{array} \approx 1500e^{0.05(4 - t_k)}\Delta t$$

The future value of the entire income stream is the sum of the future values of the money deposited during each of the $n$ subintervals. Hence,

$$\text{Future value of income stream} \approx \sum^{n} 1500e^{0.05(4 - t_k)}\Delta t$$

$k = 1$

(Note that this is only an approximation because it is based on the assumption that all $1500\Delta t_k$ dollars are deposited at time $t_k$ rather than continuously throughout the $k$th subinterval.)

As $n$ increases without bound, the length of each subinterval approaches zero and the approximation approaches the true future value of the income stream. Hence,

$$\text{Future value of income stream} = \lim_{n\to+\infty} \sum_{k=1}^{n} 1500e^{0.05(4-t_k)}\Delta t$$

$$= \int_0^4 1500e^{0.05(4-t)}dt$$

$$= 1500e^{0.2}\int_0^4 e^{-0.05t}dt$$

$$= -\frac{1500}{0.05}e^{0.2}(e^{-0.05t})\Big|_0^4$$

$$= -30000e^{0.2}(e^{-0.2} - 1)$$

$$= -30000 + 30000e^{0.2}$$

$$\approx \$6642.08$$

Correct answer is: 6642.08

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2.

**Money is transferred continuously into an account at the constant rate of \$2400 per year. The account earns interest at the annual rate of 4% compounded continuously. How much will be in the account at the end of 4 years? Round your final answer to the nearest cent.**

**The future value of the income stream is \$ ____ .**

Solution:

Note that $P$ dollars invested at 4% compounded continuously will be worth $Pe^{0.04t}$ dollars $t$ years later.

To approximate the future value of the income stream, divide the 4−year time interval

$0 \le t \le 4$ into $n$ equal subintervals of length $\Delta t$ years and let $t_k$ denote the beginning of the $k$th subinterval. Then, during the $k$th subinterval (of length $\Delta t$ years),

$$\text{Money deposited} = \text{(dollars per year)(number of years)} = 2400\Delta t$$

If all of this money were deposited at the beginning of the subinterval (at time $t_k$), it would remain in the account for $4 - t_k$ years and therefore would grow to $(2400\Delta t)e^{0.04(4 - t_k)}$ dollars. Thus,

$$\begin{matrix}\text{Future value of}\\ \text{money deposited}\\ \text{during } k\text{th subinterval}\end{matrix} \approx 2400e^{0.04(4 - t_k)}\Delta t$$

The future value of the entire income stream is the sum of the future values of the money deposited during each of the $n$ subintervals. Hence,

$$\text{Future value of income stream} \approx \sum_{k=1}^{n} 2400e^{0.04(4 - t_k)}\Delta t$$

(Note that this is only an approximation because it is based on the assumption that all $2400\Delta t_k$ dollars are deposited at time $t_k$ rather than continuously throughout the $k$th subinterval.)

As $n$ increases without bound, the length of each subinterval approaches zero and the approximation approaches the true future value of the income stream. Hence,

$$\begin{aligned}\text{Future value of income stream} &= \lim_{n\to+\infty} \sum_{k=1}^{n} 2400e^{0.04(4 - t_k)}\Delta t\\ &= \int_0^4 2400e^{0.04(4 - t)}dt\\ &= 2400e^{0.16}\int_0^4 e^{-0.04t}dt\\ &= -\frac{2400}{0.04}e^{0.16}(e^{-0.04t})\Big|_0^4\\ &= -60000e^{0.16}(e^{-0.16} - 1)\\ &= -60000 + 60000e^{0.16}\\ &\approx \$10410.65\end{aligned}$$

Correct answer is: 10410.65

3.

**Jane is trying to decide between two investments. The first costs $1000 and is expected to generate a continuous income stream at the rate of $f_1(t) = 3500e^{0.03t}$ dollars per year. The second investment costs $4300 and is estimated to generate income at the constant rate of of $f_2(t) = 4200$ dollars per year. If the prevailing annual interest rate remains fixed at 5% compounded continuously over the next 5 years, which investment will generate more net income over this time period? Round your intermediate calculations to two decimal places.**

**A. The second investment is better.**

**B. The first investment is better.**

Solution:

The net income generated by each investment over the 5−year time period is the present value of the investment less its initial cost. For each investment, we have $r = 0.05$ and $T = 5$.

For the first investment:

$$\begin{aligned} PV - \text{cost} &= \int_0^5 (3500e^{0.03t})e^{-0.05t}\,dt - 1000 \\ &= 3500\int_0^5 e^{0.03t - 0.05t}\,dt - 1000 \\ &= 3500\int_0^5 e^{-0.02t}\,dt - 1000 \\ &= 3500\left(\frac{e^{-0.02t}}{-0.02}\right)\Bigg|_0^5 - 1000 \\ &= -175000[e^{-0.02(5)} - e^0] - 1000 \\ &= 15653.45 \end{aligned}$$

For the second investment:

$$PV - \text{cost} = \int_0^5 (4200)e^{-0.05t}\,dt - 4300$$

$$= 4200\left(\frac{e^{-0.05t}}{-0.05}\right)\Bigg|_0^5 - 4300$$

$$= -84000[e^{-0.05(5)} - e^0] - 4300$$

$$= 14280.73$$

Thus, the net income generated by the first investment is \$15653.45, while the second investment generates net income of \$14280.73. The first investment is better.

Correct answer is: B

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4.

**Find the Gini index for the Lorentz curve $L(x) = 0.84x^2 + 0.16x$.**

**Gini index = ____**

Solution:

The Gini index is given by the formula

$$\text{Gini index} = 2\int_0^1 [x - L(x)]\,dx$$

$$= 2\int_0^1 [x - (0.84x^2 + 0.16x)]\,dx$$

$$= 2\left[-0.84\left(\frac{x^3}{3}\right) + 0.84\left(\frac{x^2}{2}\right)\right]\Bigg|_0^1$$

$$= 0.28$$

Correct answer is: 0.28

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5.

**A governmental agency determines that the Lorentz curves for the distribution of income for dentists and contractors in a certain state are given by the functions**

$L_1(x) = x^{2.7}$ **and** $L_2(x) = 0.2x^2 + 0.8x$

**respectively. For which profession is the distribution of income more fairly distributed?**

**A. Contractors**

**B. Dentists**

Solution:

The respective Gini indices are

$$G_1 = 2\int_0^1 (x - x^{2.7})dx = 2\left(\frac{x^2}{2} - \frac{x^{3.7}}{3.7}\right)\Bigg|_0^1 = 0.4595$$

and

$$G_2 = 2\int_0^1 [x - (0.2x^2 + 0.8x)]dx$$

$$= 2\left[-0.2\left(\frac{x^3}{3}\right) + 0.2\left(\frac{x^2}{2}\right)\right]\Bigg|_0^1 = 0.0667$$

Since the Gini index for contractors is smaller, it follows that in this state, the incomes of contractors are more evenly distributed than those of dentists.

Correct answer is: A