Math 525, Solutions 1

Due Fri., September 19, 2008

Hatcher Chapter 0, Question 18

Claim: $S^m * S^n = S^{m+n+1}$

Proof: Define $f\colon S^m \times S^n \times I \subset \mathbb{R}^{n+1} \times \mathbb{R}^{m+1} \times \mathbb{R} \to S^{n+m+1} \subset \mathbb{R}^{n+m+2} = \mathbb{R}^{n+1} \times \mathbb{R}^{m+1}$ by

$$f(x, y, t) = \frac{1}{\sqrt{t^2 + (1-t)^2}}(tx, (1-t)y).$$

Since $|x| \neq 0$ for all $x \in S^m$ and $|y| \neq 0$ for all $y \in S^n$, this map is well-defined. Since f is closed, it is a quotient map. It collapses $S^m \times \{y\} \times \{0\}$ to a single point for all $y \in S^n$, and similarly collapses $\{x\} \times S^n \times \{1\}$ to a point for all $x \in S^m$; otherwise the map is one-to-one. Finally the map is surjective; fix $(u, v) \in S^{n+m+1}$. If $v = 0$, then $f(u, v, 1) = (u, v)$. If $u = 0$, then $f(u, v, 0) = (u, v)$ Otherwise, $f(\frac{u}{|u|}, \frac{v}{|v|}, \frac{|u|}{|u|+|v|}) = (u, v)$.

Hatcher Chapter 0, Question 19

Claim: Let X be a space obtained from S^2 by attaching n 2-cells along any collection of n circles in S^2. Then X is homotopy equivalent to a wedge sum of $n+1$ 2-spheres.

Warning: I am assuming the attaching maps are *embeddings*.

Proof: Clearly $(\coprod D^2, \coprod S^1)$ is a CW pair. Moreover, any embedding $f\colon S^1 \to S^2$ must miss at least one point $x \in S^2$. Therefore, the homotopy equivalence between $S^2 \setminus \{x\} = \mathbb{R}^2$ and a point induces a homotopy from f to the constant map x_0. So the result follows from Proposition 0.18.