Balloons, Cut-Edges, Matchings, and Total Domination in Regular Graphs of Odd Degree

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Abstract

A *balloon* in a graph G is a maximal 2-edge-connected subgraph incident to exactly one cut-edge of G. Let $b(G)$ be the number of balloons, let $c(G)$ be the number of cut-edges, and let $\alpha'(G)$ be the maximum size of a matching. Let $\mathcal{F}_n$ be the family of connected $(2r+1)$-regular graphs with n vertices. For $G \in \mathcal{F}_n$, we prove $c(G) \le \frac{r(n-2)-2}{2r^2+2r-1} - 1$ and $\alpha'(G) \ge \frac{n}{2} - \frac{rb(G)}{2r+1}$. Also $b(G) \le \frac{(2r-1)n+2}{4r^2+4r-2}$, which yields a simple proof of the lower bound on $\alpha'(G)$ by Henning and Yeo (about $\frac{n}{2} - \frac{n}{4r}$ for large r). For each of these bounds and each r, we determine the infinite family where equality holds. For the total domination number $\gamma_t(G)$ of a cubic graph, we prove $\gamma_t(G) \le \frac{n}{2} - \frac{b(G)}{2}$ (except that $\gamma_t(G)$ may be $n/2-1$ when $b(G) = 3$ and the balloons cover all but one vertex). With $\alpha'(G) \ge \frac{n}{2} - \frac{b(G)}{3}$ for cubic graphs, this improves the known inequality $\gamma_t(G) \le \alpha'(G)$.

1 Introduction

A graph is a *cubic graph* if every vertex has degree 3. In 1891, Petersen [10] proved that every cubic graph without cut-edges has a perfect matching. It is natural to ask how small $\alpha'(G)$ can be in a cubic graph G with n vertices, where $\alpha'(G)$ is the maximum size of a matching in G (called the *matching number* of G). Chartrand et al. [3] proved that $\alpha'(G) \ge n/2 - \lceil c(G)/3 \rceil$ when G is a cubic n-vertex graph, where $c(G)$ denotes the number of cut-edges in G.

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By this result, an upper bound on $c(G)$ yields a lower bound on $\alpha'(G)$. Let G be a connected cubic graph with n vertices. In Section 3, we prove that $c(G) \le (n-7)/3$ and that this is sharp. The result of [3] then yields $\alpha'(G) \ge (7n+14)/18$, but this is not the best bound on $\alpha'(G)$. The smallest value of $\alpha'(G)$ is $\lceil (4n-1)/9 \rceil$, proved first by Biedl et al. [2]. Henning and Yeo [8] generalized the result, proving that $\alpha'(G) \ge \frac{n}{2} - \frac{r}{2}\frac{(2r-1)n-1}{(2r+1)(2r^2+2r-1)}$ when G is a $(2r+1)$-regular n-vertex connected graph, which is sharp.

Although maximizing $c(G)$ in a cubic graph does not minimize $\alpha'(G)$, another concept does yield a simple proof of the sharp bound on $\alpha'(G)$. We define a *balloon* in a graph G to be a maximal 2-edge-connected subgraph of G incident to exactly one cut-edge of G. The term arises from viewing the cut-edge as a string tying the balloon to the rest of the graph; the vertex incident to the cut-edge is the *neck* of the balloon. A balloon may contain cut-vertices and thus consist of several blocks.

Maximal 2-edge-connected subgraphs are pairwise disjoint, since the union of two 2-edge-connected subgraphs sharing a vertex is also 2-edge-connected. Among these subgraphs, the balloons are those incident to precisely one cut-edge. Thus the number of balloons in G is well-defined; let $b(G)$ denote this number.

Let $\mathcal{F}_n$ be the family of connected $(2r+1)$-regular graphs with n vertices. For $G \in \mathcal{F}_n$, we prove that $c(G) \le \frac{r(n-2)-2}{2r^2+2r-1} - 1$ and $\alpha'(G) \ge \frac{n}{2} - \frac{rb(G)}{2r+1}$. We obtain a lower bound on $\alpha'(G)$ by proving that $b(G) \le \frac{(2r-1)n+2}{4r^2+4r-2}$, and we use balloons to prove the upper bound on $c(G)$. In Section 2, we construct an infinite family $\mathcal{H}_r$ showing that all these bounds are sharp; it contains the smaller families provided in [2] and [8] (graphs in $\mathcal{H}_r$ exist when $n \equiv 4(r+1)^2 \mod (8r^3+12r^2-2)$). The bounds for $b(G)$ and $c(G)$ are sharp in a larger family $\mathcal{H}'_r$ (occurring when $n \equiv (4r+6) \mod (4r^2+4r-2)$). In Section 3, we prove the upper bounds on $b(G)$ and $c(G)$ and show that equality holds if and only if $G \in \mathcal{H}'_r$. In Section 4, we prove the lower bound on $\alpha'(G)$; in Section 5, we show that equality holds if and only if $G \in \mathcal{H}_r$.

The restriction to connected graphs is important; consider cubic graphs. For a connected cubic graph, $b(G) \le (n+2)/6$ and $\alpha'(G) \ge (4n-1)/9$. However, if G consists of many disjoint copies of the unique 16-vertex cubic graph with no perfect matching, then $b(G) = 3n/16$ and $\alpha'(G) = 7n/16$; these values are more extreme than the bounds for graphs in $\mathcal{F}_n$.

In Section 6, we use balloons to study total domination. A *total dominating set* in a graph G is a set S of vertices in G such that every vertex in G has a neighbor in S. The *total domination number*, written $\gamma_t(G)$, is the minimum size of a total dominating set in G. Henning, Kang, Shan, and Yeo [6] proved that $\gamma_t(G) \le \alpha'(G)$ for every regular graph G with degree at least 3. For degree at least 4, stronger bounds hold. Thomassé and Yeo [11] proved that $\gamma_t(G) \le 3n/7$ for every n-vertex regular graph with degree at least 4. This upper bound is a smaller fraction of n than the lower bound on $\alpha'(G)$. Earlier, Henning and

Yeo [8] observed that $\gamma_t(G) < \alpha'(G)$ when G is a regular graph with degree at least 4.

We use balloons to strengthen the bound for cubic graphs. We prove that $\gamma_t(G) \le \frac{n}{2} - \frac{b(G)}{2}$ when G is cubic, except that $\gamma_t(G) = n/2 - 1$ is possible when $b(G) = 3$ and the balloons cover all but one vertex. Since $\alpha'(G) \ge \frac{n}{2} - \frac{b(G)}{3}$ for cubic graphs, we have large separation when the number of balloons is large, and $\gamma_t(G) = \alpha'(G)$ can happen in a cubic graph only when the number of balloons is 0 or when G consists of three balloons plus one vertex.

We mention one related result. The extension of Petersen's result by Bäbler [1] states that every $(2r+1)$-regular $2r$-edge-connected graph has a perfect matching. As the edge-connectivity rises, the lower bound on the matching number should also rise. We solve this problem in a subsequent paper [9], determining the smallest matching number for d-regular k-edge-connected graphs with n vertices, when $d \ge 4$ and $k \ge 2$. The proof differs somewhat from the techniques in this paper, since k-edge-connected graphs have no balloons. A generalization of balloons is needed.

2 The Construction

Biedl et al. [2] and Henning and Yeo [8] presented examples for sharpness in the lower bounds on $\alpha'(G)$ for connected 3-regular and $(2r+1)$-regular graphs, respectively. We present a more general family that includes their examples.

Construction 2.1. Let B_r be the graph obtained from the complete graph K_{2r+3} by deleting a matching of size $r+1$ and one more edge incident to the remaining vertex. This is the smallest graph in which one vertex has degree $2r$ and the others have degree $(2r+1)$. Thus B_r is the smallest possible balloon in a $(2r+1)$-regular graph. Note that deleting the vertex of degree $2r$ (the neck) from B_r leaves a subgraph having a perfect matching.

Let $\mathcal{T}'_r$ be the family of trees such that every non-leaf vertex has degree $2r+1$. Let $\mathcal{H}'_r$ be the family of $(2r+1)$-regular graphs obtained from trees in $\mathcal{T}'_r$ by identifying each leaf of such a tree with the neck in a copy of B_r. Let $\mathcal{T}_r$ be the subfamily of $\mathcal{T}'_r$ obtained by requiring all leaves to have the same color in a proper 2-coloring. Let $\mathcal{H}_r$ be the subfamily of $\mathcal{H}'_r$ arising from trees in $\mathcal{T}_r$ by adding balloons at leaves. □

To compute the matching number for n-vertex graphs in $\mathcal{H}_r$, we use standard concepts about matchings. The *deficiency* of a vertex set S in a graph G, written $\mathrm{def}_G(S)$ or simply $\mathrm{def}(S)$, is $o(G-S) - |S|$, where $o(H)$ is the number of components of H having an odd number of vertices. Every matching must leave at least $\mathrm{def}(S)$ vertices unmatched, so for any S the quantity $\frac{1}{2}(n - \mathrm{def}(S))$ is an upper bound on $\alpha'(G)$. Furthermore, if there is a matching that matches S into vertices of distinct odd components of $G-S$ and leaves at most one unmatched vertex in each odd component of $G-S$, then $\alpha'(G) = \frac{1}{2}(n - \mathrm{def}(S))$.

Proposition 2.2. *Let $p_r = 2r^2 + 2r - 1$. For any n-vertex graph G in $\mathcal{H}_r$,*

$$n \equiv 4(r+1)^2 \mod ((4r+2)p_r), \qquad b(G) = \frac{(2r-1)n+2}{2p_r},$$
$$\alpha'(G) = \frac{1}{2}\left(n - \frac{r(2r-1)n+2r}{(2r+1)p_r}\right), \qquad c(G) = \frac{r(n-2)-2}{p_r} - 1.$$

Furthermore, the formulas given for $b(G)$ and $c(G)$ also hold when $G \in \mathcal{H}'_r$.

Proof. We first compute $b(G)$ and $c(G)$ on $\mathcal{H}'_r$. The smallest tree in $\mathcal{T}'_r$ has two vertices. The resulting graph in $\mathcal{H}'_r$ has $4r+6$ vertices, two balloons, and one cut-edge, and the formulas hold. For any larger tree T in $\mathcal{T}'_r$, the penultimate vertex of a longest path has $2r$ leaf neighbors, and deleting them yields a smaller tree T' in $\mathcal{T}'_r$. Let G and G' be the corresponding graphs in $\mathcal{H}'_r$. Compared to G', in G there are $2r$ more cut-edges, $2r-1$ more balloons, and $2r(2r+3) - (2r+2)$ more vertices. This last formula simplifies to $2p_r$, and hence the formulas for $b(G)$ and $c(G)$ in terms of n are established by induction on n.

Now consider the more restrictive families $\mathcal{T}_r$ and $\mathcal{H}_r$. The smallest graph in $\mathcal{T}_r$ is the star $K_{1,2r+1}$ with $2r+1$ leaves. We claim that every other tree in $\mathcal{T}_r$ arises from a smaller tree in $\mathcal{T}_r$ by appending $2r$ edges at a leaf y and appending $2r$ additional edges at each new neighbor of y. This produces $(2r)^2$ leaves, which replace y in the set of leaves and are in the same partite set as y, so the larger graph lies in $\mathcal{T}_r$.

To prove that this generates all of $\mathcal{T}_r$, consider a longest path P in a tree $T \in \mathcal{T}_r$ such that T is not a star. Let y, z, w be the last three vertices on P, in order (w is the leaf). Since P is a longest path, all $2r$ neighbors of z other than y are leaves. Since leaves all lie in the same partite set, no neighbor of y is a leaf. Hence the $2r-1$ neighbors of y not on P must all have $2r$ leaf neighbors (again since P is a longest path and non-leaves have degree $2r+1$). Now T arises in the specified way from a smaller tree in $\mathcal{T}_r$ having y as a leaf.

To compute $\alpha'(G)$ for $G \in \mathcal{H}_r$, let T be the corresponding tree in $\mathcal{T}_r$. Let X and Y be its partite sets, with Y containing the leaves. Let $S = X$. Now $o(G-S) = |Y|$, since each vertex of Y is an isolated vertex in $G-S$ or is the neck of a copy of B_r that is an odd component of $G-S$. Thus $\mathrm{def}(S) = |Y| - |X|$. Root T at a vertex of X, and then match each vertex of S to one of its children, which is or lies in an odd component of $G-S$. When that odd component is a copy of B_r, pair its remaining vertices in a matching. This produces a matching with exactly $\mathrm{def}(S)$ uncovered vertices.

It therefore suffices to compare $\mathrm{def}(S)$ and the formula for $\alpha'(G)$ inductively. When $T = K_{1,2r+1}$, we have $\mathrm{def}(S) = 2r$. Adding the balloons yields $(2r+3)(2r+1)+1$ (this equals $4(r+1)^2$, giving the basis for the claim about n). The subtractive term in the formula for $\alpha'(G)$ is $\frac{r(2r-1)(4r^2+8r+4)+2r}{(2r+1)p_r}$, which equals $2r$.

For larger $G \in \mathcal{H}_r$, let T be the corresponding tree in $\mathcal{T}_r$, expanded from T' with corresponding graph $G' \in \mathcal{H}_r$. In the expansion, $|X|$ increases by $2r$ and $|Y|$ increases by $4r^2$, so def(S) increases by $4r^2-2r$. Comparing G with G', one balloon is lost and $4r^2$ are created; the number of vertices increases by $4r^2(2r+3)+2r-(2r+2)$. The increase in n simplifies to $(4r+2)p_r$ (completing the proof of the claim about n). The subtractive term in the formula for $\alpha'(G)$ thus increases by $r(2r-1)2$, which equals the change in def(S). $\square$

Corollary 2.3. *For n-vertex cubic graphs, the matching number of graphs in $\mathcal{H}_1$ is $\frac{4n-1}{9}$.*

3 Balloons and Cut-edges

Recall that $\mathcal{F}_n$ is the family of connected $(2r+1)$-regular graphs with n vertices. We begin by bounding the number of balloons for graphs in $\mathcal{F}_n$.

Every balloon in a $(2r+1)$-regular graph has at least $2r+3$ vertices; it has at least $2r+2$ vertices because it has a vertex of degree $2r+1$, and equality cannot hold because then the degree-sum would be odd. Thus $b(G) \le \frac{n}{2r+3}$. Surprisingly, this trivial upper bound can be improved only slightly; the optimal bound is $\frac{n+2\epsilon}{2r+3+\epsilon}$, where $\epsilon = 1/(2r-1)$. Of course, $\epsilon = 1$ for cubic graphs. We use a counting argument; the bound can also be proved inductively.

Lemma 3.1. *If $G \in \mathcal{F}_n$, then $b(G) \le \frac{(2r-1)n+2}{4r^2+4r-2}$, with equality if and only if $G \in \mathcal{H}'_r$.*

Proof. For $G \in \mathcal{F}_n$, let G' be the graph obtained from G by shrinking each balloon to a single vertex; G' is connected, and the balloons of G become vertices of degree 1 in G'. Let $n' = |V(G')|$ and $m' = |E(G')|$. Since G' is connected, $m' \ge n'-1$, and the degree-sum formula yields $(2r+1)n' - 2rb(G) = 2m' \ge 2n'-2$. Thus $2rb(G) \le (2r-1)n'+2$. Since each balloon has at least $2r+3$ vertices, $n' \le n-(2r+2)b(G)$. Combining the inequalities yields $2rb(G) \le (2r-1)n+2-(2r-1)(2r+2)b(G)$, which simplifies to the desired bound.

Equality requires equality in each contributing inequality. Hence G' is a tree with non-leaf vertices having degree $2r+1$. That is, $G' \in \mathcal{T}'_r$, and $G \in \mathcal{H}'_r$. $\square$

Corollary 3.2. *Every connected n-vertex cubic graph has at most $\frac{n+2}{6}$ balloons, and this is sharp for $n \equiv 4 \mod 6$.*

The bounds of Lemma 3.1 and Corollary 3.2 do not hold for disconnected graphs. An n-vertex graph consisting of disjoint copies of the smallest graph in $\mathcal{H}_r$ has $\frac{2r+1}{6r+10}n$ balloons, which is more than the bound above.

Lemma 3.3. *The following hold for balloons and cut-edges in graphs in $\mathcal{F}_n$.*
(a) Each component formed by deleting a cut-edge contains a balloon.
(b) Balloons may have any odd number of vertices at least $2r+3$.

Proof. (a) Let e be a cut-edge. Among the paths containing e, let P be a path containing the maximum number of cut-edges of G. The portion of P after the last cut-edge toward either end lies in a 2-edge-connected subgraph, and by the choice of P it is a balloon.

(b) In a balloon, the neck has degree $2r$, and other vertices have degree $2r+1$. Such graphs exist with every odd number of vertices at least $2r+3$. For $k \geq r$, the complete graph K_{2k+3} decomposes into $k+1$ spanning cycles. The union of r of these cycles plus a near-perfect matching from one of the remaining cycles is a 2-edge-connected graph with the desired degrees. □

Lemma 3.4. *If $G \in \mathcal{F}_n$, then $c(G) \leq \frac{r(n-2)-2}{2r^2+2r-1} - 1$, with equality if and only if $G \in \mathcal{H}'_r$.*

Proof. We use induction on n. If $n \leq 4r+6$, then the bound at most 1, with equality only when $n = 4r+6$. Every graph having a cut-edge has at least two balloons and hence at least $4r+6$ vertices, by Lemma 3.3. The graph with $4r+6$ vertices consisting of two copies of B_r joined by an edge lies in $\mathcal{H}'_r$. Hence all claims hold for the basis.

For larger n, consider a cut-edge e in G. Let G_1 and G_2 be the components of $G-e$. Let G'_1 and G'_2 be the graphs obtained from G by replacing G_2 and G_1, respectively, with B_r. The cut-edges of G consists of the cut-edges in G_1 and G_2, plus e itself. Since e is a cut-edge in both G'_1 and G'_2, and the added B_r contains no cut-edge, we have $c(G) = c(G'_1) + c(G'_2) - 1$. If neither G_1 nor G_2 equals B_r, then G'_1 and G'_2 have fewer vertices than G, and we can apply the induction hypothesis to both. Letting $n_i = |V(G'_i)|$, we have $n = n_1 + n_2 - (4r+6)$. With $p_r = 2r^2 + 2r - 1$ (as in Proposition 2.2), we obtain the desired bound on $c(G)$:

$$\begin{aligned} c(G) &= c(G'_1) + c(G'_2) - 1 \leq \frac{r(n_1-2)-2}{p_r} + \frac{r(n_2-2)-2}{p_r} - 3 \\ &= \frac{r(n-2)-2}{p_r} + \frac{r(4r+4)-2}{p_r} - 3 = \frac{r(n-2)-2}{p_r} - 1. \end{aligned}$$

In the remaining case, every cut-edge in G is incident to a copy of B_r. Since each copy of B_r is incident to exactly one cut-edge, we obtain $c(G) = b(G)$ (note that $n > 4r+6$). Let Q be the set of endpoints of cut-edges outside the balloons. If any two balloons have distinct nonadjacent neighbors in Q, then let G' be the graph obtained by deleting the two balloons and adding one edge to make their neighbors adjacent. The graph G' is connected and $(2r+1)$-regular and has $n-(4r+6)$ vertices. Crucially, G' has exactly $c(G)-2$ cut-edges, because the only cut-edges in G are those incident to balloons. By the induction hypothesis,

$$c(G) \leq 2 + \frac{r(n-4r-8)-2}{p_r} - 1 = \frac{r(n-2)-2}{p_r} - \frac{4r^2+6r}{p_r} + 1 < \frac{r(n-2)-2}{p_r} - 1.$$

Hence we may assume that the vertices of Q are pairwise adjacent. Let $q = |Q|$, and let S

be the set of vertices outside both Q and the balloons. If $S = \varnothing$, then $c(G) = q(2r+2-q)$ and $n = (2r+3)c(G)+q$. Since $1 \le q \le 2r+1$, we obtain $n \ge (2r+3)(2r+1)+1 = 2p_r+4r+6$. Since $c(G) = b(G)$, Lemma 3.1 yields $c(G) \le \frac{(2r-1)n+2}{2p_r} = \frac{rn}{p_r} - \frac{n-2}{2p_r}$. It thus suffices to show that $\frac{n-2}{2p_r} \ge \frac{2r+2}{p_r} + 1$. This requires $n-2 \ge 4r+4+2p_r$, which we have proved for this case.

Finally, suppose that $S \neq \varnothing$. Each vertex of S has $2r+1$ neighbors outside the balloons, so $n \ge 2r+2+(2r+3)c(G)$. If equality holds, then $S \cup Q$ induces a complete graph, $G = K_{r+2}$, and $c(G) = 0$. Otherwise, $n \ge (2r+3)[c(G)+1]$. Now $c(G) \le \frac{n}{2r+3} - 1$, and we only need $\frac{n}{2r+3} \le \frac{r(n-2)-2}{2r^2+2r-1}$. This simplifies to $n \ge 4r+6$, which holds when $c(G) > 0$.

For the characterization of equality, consider each case. When G has a cut-edge not incident to a balloon that is a copy of B_r, the induction hypothesis requires achieving equality for both G_1' and G_2', which must therefore lie in $\mathcal{H}_r'$. The construction of G from G_1' and G_2' indeed puts G in $\mathcal{H}_r'$. When $c(G) = b(G)$ and two balloons have nonadjacent neighbors, we obtained strict inequality in the bound. When $c(G) = b(G)$ and $S = \varnothing$, equality requires $b(G)$ to meet its bound, which already requires $G \in \mathcal{H}_r'$ (indeed, it requires more, and equality is obtained only by putting copies of B_r at the leaves of the star $K_{1,2r+1}$). When $S \neq \varnothing$, equality requires $n = 4r+6$ and $c(G) = 1$, in which case G is the graph in $\mathcal{H}_r'$ consisting of a cut-edge joining two copies of B_r. □

Corollary 3.5. *Every n-vertex $(2r+1)$-regular graph has at most $\frac{r(n-2)-2}{2r^2+2r-1} - 1$ cut-edges, which reduces to $\frac{n-7}{3}$ for cubic graphs.*

Proof. Since the contributions not linear in n are negative and we seek an upper bound, the bound holds also for disconnected n-vertex $(2r+1)$-regular graphs. □

4 Balloons and Matchings

Here we use balloons to prove the result of Henning and Yeo [8] minimizing the matching number for n-vertex $(2r+1)$-regular connected graphs; in the next section we characterize the graphs where equality holds.

We use the Berge–Tutte Formula for the matching number. Recall that the deficiency $\mathrm{def}(S)$ of a vertex set S in G is defined by $\mathrm{def}(S) = o(G-S) - |S|$. Tutte [12] proved that a graph G has a 1-factor if and only if $\mathrm{def}(S) \le 0$ for all $S \in V(G)$. The equivalent Berge–Tutte Formula (see Berge [1]) states that $\alpha'(G) = \min_{S \subseteq V(G)} \frac{1}{2}(n - \mathrm{def}(S))$.

Lemma 4.1. *Let G be an n-vertex $(2r+1)$-regular graph, and let S be a subset of $V(G)$. If the number of edges from each odd component of $G-S$ to S is only 1 or is at least $2r+1$, then* $\mathrm{def}(S) \le \frac{2rb(G)}{2r+1}$.

Proof. Let c_1 be the number of odd components of $G - S$ having one edge to S. By Lemma 3.3(a), each component of $G - S$ having one edge to S contains a balloon. Thus $c_1 \le b(G)$. Counting the edges joining S to odd components of $G - S$ yields

$$(2r + 1)|S| \ge (2r + 1)o(G - S) - 2rc_1 \ge (2r + 1)o(G - S) - 2rb(G),$$

and hence $\mathrm{def}(S) = o(G - S) - |S| \le \frac{2rb(G)}{2r+1}$. □

Corollary 4.2. *If G is a connected cubic graph, then $\alpha'(G) \ge \frac{n}{2} - \left\lfloor \frac{b(G)}{3} \right\rfloor$.*

Proof. In a 3-regular graph, all edge-cuts between sets of odd size have odd size, which is 1 or at least 3. Hence Lemma 4.1 yields the claim (using the floor function in the second term is valid because $\alpha'(G)$ and $n/2$ are integers). □

If in a connected graph G some set of maximum deficiency satisfies the hypothesis of Lemma 4.1, then $\alpha'(G) \ge \frac{n}{2} - \frac{r}{2}\frac{(2r-1)n+2}{(2r+1)(2r^2+2r-1)}$, by the Berge–Tutte Formula and Lemma 3.1. We prove this bound for all connected odd-regular graphs and determine the extremal graphs.

Theorem 4.3. *If $G \in \mathcal{F}_n$, then $\alpha'(G) \ge \frac{n}{2} - \frac{r}{2}\frac{(2r-1)n+2}{(2r+1)(2r^2+2r-1)}$, with equality if and only if $G \in \mathcal{H}_r$.*

Proof. By the Berge–Tutte Formula, it suffices to show that every set $S \subseteq V(G)$ has deficiency at most $r\frac{(2r-1)n+2}{(2r+1)(2r^2+2r-1)}$. By Lemma 4.1, we may assume that there is an odd component of $G - S$ such that the number of edges from $G - S$ to S is between 3 and $2r - 1$; call such a component of $G - S$ a *bad subgraph*.

For each edge e joining S to a bad subgraph, replace e with a cut-edge incident to a copy of B_r at its end outside S. Also delete all vertices in bad subgraphs. Let G' denote the resulting graph; note that G' is $(2r + 1)$-regular. Unfortunately, G' may be disconnected.

Let c be the number of bad subgraphs, and let x be the total number of vertices in them. Let y be the total number of edges in G joining S to bad subgraphs; y is the number of balloons added in forming G'.

Let p be the number vertices in some bad subgraph Q. If $p \le 2r + 1$, then regularity forces each vertex of Q to have at least $2r+2-p$ neighbors in S. Hence the number of edges from S to $V(Q)$ is at least $p(2r + 2 - p)$, which is at least $2r + 1$, contradicting that Q is a bad subgraph. We conclude that $p \ge 2r + 3$, and hence $x \ge (2r + 3)c$.

The number of vertices in G' is $n-x+(2r+3)y$. We also need the number of components of G'. Each time we pull an edge off a bad subgraph Q and make it incident to a copy of B_r, we increase the number of components by 0 or 1. Doing this with the last edge to Q (and deleting $V(Q)$) does not change the number of components. Since G is connected, we conclude that G' has at most $1 + y - c$ components.

The alteration from G to G' ensures that S satisfies the hypotheses of Lemma 4.1 for G'. Lemma 4.1 does not require connected graphs, so $\mathrm{def}_{G'}(S) \le \frac{2rb(G')}{2r+1}$. However, applying Lemma 3.1 to replace the number of balloons with upper bounds in terms of the number of vertices does require connected graphs. Therefore, we apply Lemma 3.1 to each component of G'. We obtain an additive constant 2 in the numerator for each component. Thus $b(G') \le \frac{(2r-1)(n-x+(2r+3)y)+2(1+y-c)}{4r^2+4r-2}$. With $x \ge (2r+3)c$, we have $b(G') \le \frac{(2r-1)n+2}{4r^2+4r-2} + \frac{4r^2+4r-1}{4r^2+4r-2}(y-c)$.

Meanwhile, we must also relate $\mathrm{def}_{G'}(S)$ to $\mathrm{def}_G(S)$. We have replaced c odd components in $G - S$ with y odd components in $G' - S$. Thus

$$\begin{aligned} \mathrm{def}_G(S) &= \mathrm{def}_{G'}(S) - (y-c) \le \frac{2rb(G')}{2r+1} - (y-c) \\ &\le \frac{r}{2r+1}\frac{(2r-1)n+2}{2r^2+2r-1} + \frac{2r}{2r+1}\frac{4r^2+4r-1}{4r^2+4r-2}(y-c) - (y-c) \end{aligned}$$

Thus it suffices to show that $2r(4r^2+4r-1) \le (2r+1)(4r^2+4r-2)$. This inequality has the form $ab \le (a+1)(b-1)$ with $a < b$, and hence it holds. □

Corollary 4.4. *If G is a connected n-vertex cubic graph, then $\alpha'(G) \ge \frac{4n-1}{9}$, and this is sharp infinitely often.*

5 Characterization of Equality

We proved in Proposition 2.2 that equality holds in the bound of Theorem 4.3 when $G \in \mathcal{H}_r$. Now we show that these are the only graphs achieving equality. Recall that $\mathcal{T}_r$ is the family of trees from which graphs in $\mathcal{H}_r$ are formed by appending small balloons at leaves.

Lemma 5.1. *If T is an n-vertex tree in which every non-leaf vertex has degree $2r+1$, then $\alpha'(T) \ge \frac{n-1}{2r+1}$, with equality only when $T \in \mathcal{T}_r$.*

Proof. Since T has $n-1$ edges and maximum degree $2r+1$, the number of vertices needed to cover $E(T)$ is at least $\frac{n-1}{2r+1}$, and hence the König–Egerváry Theorem yields $\alpha'(T) \ge \frac{n-1}{2r+1}$.

If all leaves lie in the same partite set, then the other partite set is a vertex cover of size $\frac{n-1}{2r+1}$. Conversely, equality holding requires a vertex cover Q of size $\frac{n-1}{2r+1}$. No two vertices of Q can cover the same edge, so Q is an independent set. Also every vertex adjacent to a leaf must be in Q, since a leaf covers only one edge.

To show that all leaves are in the same partite set, let x and y be leaves, and let P be the x,y-path in T. The edges of P must be covered by vertices on P, so Q contains a vertex of each edge of P. Since Q is independent, the vertices of P alternate between Q and not-Q, with the neighbors of x and y being in Q. Hence the distance between x and y is even, and they are in the same partite set. □

For a graph $G \in \mathcal{F}_n$ that achieves the minimum value of the matching number, we show that $G \in \mathcal{H}_r$ by showing that if we shrink each balloon to a single vertex, then the resulting graph is in $\mathcal{T}_r$.

Theorem 5.2. *If $G \in \mathcal{F}_n$ and $\alpha'(G) = \frac{n}{2} - \frac{r}{2}\frac{(2r-1)n+2}{(2r+1)(2r^2+2r-1)}$, then $G \in \mathcal{H}_r$.*

Proof. Equality in the bound requires equality in all the inequalities of Theorem 4.3. A set S with maximum deficiency must satisfy $\text{def}(S) = \frac{r}{2r+1}\frac{(2r-1)n+2}{2r^2+2r-1}$. Since the coefficient on $y-c$ in the final displayed inequality for Theorem 4.3 is negative, we must have $y = c$. This states that the total number of edges joining S to bad subgraphs equals the number of bad subgraphs, which implies that one edge goes to each bad subgraph, and therefore they are not bad. We conclude that $y = c = 0$, and the number of edges joining S to each odd component of $G - S$ is 1 or is at least $2r+1$.

Now Lemma 4.1 applies and yields $\text{def}(S) \le \frac{2rb(G)}{2r+1}$. From Lemma 3.1, we now have

$$\frac{r}{2r+1}\frac{(2r-1)n+2}{2r^2+2r-1} \le \frac{2rb(G)}{2r+1} \le \frac{r}{2r+1}\frac{(2r-1)n+2}{2r^2+2r-1},$$

so $b(G) = \frac{(2r-1)n+2}{4r^2+4r-2}$. From the proof of Lemma 3.1, equality in the bound requires each balloon to have exactly $2r+3$ vertices.

Let G' be the graph obtained from G by shrinking each balloon to a single vertex. Let $n' = |V(G')|$ and $m' = |E(G')|$. Since each balloon has $2r+3$ vertices, we have $n = n' + (2r+2)b(G)$. Substituting this expression for n into the formula $b(G) = \frac{(2r-1)n+2}{4r^2+4r-2}$ and simplifying yields $2rb(G) = (2r-1)n' + 2$.

Contraction does not disconnect, so G' is connected. To show that G' is a tree, we count the edges. By the degree sum formula,

$$2m' = (2r+1)n' - 2rb(G) = (2r+1)n' - (2r-1)n' - 2 = 2n' - 2.$$

Finally, we show $G' \in \mathcal{T}_r$. By Lemma 5.1, it suffices to show that G' has a matching of size $\frac{n'-1}{2r+1}$. Note that $\alpha'(G') \ge \alpha'(G) - (r+1)b(G)$, and we are given $\alpha'(G) = \frac{n}{2} - \frac{rb(G)}{2r+1}$. Since $\frac{n}{2} - (r+1)b(G) = \frac{n'}{2}$ and $2rb(G) = (2r-1)n' + 2$, we conclude that $\alpha'(G') \ge \frac{n'-1}{2r+1}$. □

6 Balloons and Total Domination

Balloons also help in proving bounds on the total domination number. The results are strongest for cubic graphs. We use a lemma proved by Henning that provides a useful upper bound in nearly regular graphs. Let $\Delta(G)$ and $\delta(G)$ denote the maximum and minimum vertex degrees in a graph G.

Lemma 6.1. (Henning's Lemma [5]) *If G is a graph with n vertices and m edges, then* $\gamma_t(G) \le n - \frac{m}{\Delta(G)}$. □

Lemma 6.2. *If B is a balloon with p vertices in a cubic graph G, then $\gamma_t(B) \le \frac{p-1}{2}$. Furthermore, B has a dominating set of size $(p-1)/2$ that contains the neck of B and a neighbor of every vertex other than the neck.*

Proof. Let v be the neck of B. Recall that v has degree 2 in B, and the other vertices of B have degree 3 in B. By Henning's Lemma, $\gamma_t(B) \le p - (3p-1)/6 = p/2 + 1/6$. Since p is odd and $\gamma_t(B)$ is an integer, $\gamma_t(B) \le (p-1)/2$.

Let T be the set consisting of v and its two neighbors in B. The number of edges joining T and $V(B) - T$ is 2 or 4, depending on whether T induces a triangle. Note that $B - T$ has $p-3$ vertices and at least $[3(p-3) - 4]/2$ edges. If $\Delta(B - T) = 3$, then Henning's Lemma yields $\gamma_t(B - T) \le (p-3) - (3p-9-4)/6 = (p-3)/2 + 2/3$. Since p is odd and $\gamma_t(B - T)$ is an integer, $\gamma_t(B) \le (p-3)/2$ in this case, and adding v to a smallest total dominating set of $B - T$ yields the desired set.

In the remaining case, $\Delta(B - T) < 3$. Since deleting T removes at most four edges incident to $V(B) - T$, this case requires $p \le 7$. If $p = 7$, then $B - T = C_4$, and T is a total dominating set of size $(p-1)/2$ containing v. If $p = 5$, then B is the unique smallest balloon B_1, and v with one of its neighbors forms a total dominating set of size $(p-1)/2$. □

When $|V(B)| = 7$, it may happen that B has no total dominating set of size $(p-1)/2$ containing its neck. If the neck induces a triangle with its neighbors, then the remaining four vertices induce five edges, and no total dominating set of size 3 contains the neck. Call this special balloon $\hat{B}$.

In addition to small dominating sets, we also need large matchings in balloons.

Lemma 6.3. *Every balloon in a 3-regular graph has a matching that covers every vertex except its neck.*

Proof. Let v be the neck of a balloon B, with $N(v) = \{u, w\}$. Let B' consist of two disjoint copies of B plus a cut-edge joining their necks. Now B' is a 3-regular graph with one cut-edge, since B has no cut-edge.

Petersen proved that a 3-regular graphs with at most two cut-edges has a perfect matching. Since B' has odd order, the cut-edge lies in every perfect matching. Deleting it leaves the desired matching in B. □

Since $\alpha'(G) \ge \frac{n}{2} - \frac{b(G)}{3}$ when G is 3-regular and connected (Corollary 4.2, proving $\gamma_t(G) \le \frac{n}{2} - \frac{b(G)}{2}$ would yield $\gamma_t(G) \le \alpha'(G)$, with equality only when $b(G) = 0$. However, the desired upper bound may fail when G consists of three balloons plus one common neighbor.

The 2-edge-connected case (no balloons) has been well-studied. By Henning's Lemma, $\gamma_t(G) \le n/2$. Equality may hold when G is 2-edge-connected; such graphs were characterized by Henning, Soleimanfallah, Thomassé, and Yeo [7]. The graphs achieving equality consist of two infinite families and one additional 16-vertex graph. In one family, the graph consists of two even cycles with vertex sets $x_1, \ldots, x_{2k}$ and $y_1, \ldots, y_{2k}$, plus the edges $x_{2i-1}y_{2i}$ and $x_{2i}y_{2i-1}$ for $1 \le i \le k$. Being 2-edge-connected, these graphs also have perfect matchings, so here $\gamma_t(G) = \alpha'(G)$.

Hence we may confine our attention to graphs having balloons. Our strategy is to assemble a small total dominating set S using $(|V(B)| - 1)/2$ vertices in each balloon B and $|V(G')|/2$ vertices in the graph G' obtained by deleting the balloons. This gives the desired size. Vertices having neighbors in balloons have degree less than 3 in G'. Such a vertex in S does not need a neighbor in $S \cap V(G')$; Lemma 6.2 allows us to give it the neck of the balloon as a neighbor. This weakened restriction on S as a dominating set in G' motivates the following definition.

Definition 6.4. *A dominating set S in a graph G is a* semitotal dominating set *(abbreviated* SD-set*) if every vertex with maximum degree in G has a neighbor in S.*

In an SD-set, vertices of non-maximum degree can dominate themselves. The problem of finding an SD-set, like the problem of finding a total dominating set, can be modeled using hypergraphs. In the generalization of graphs to hypergraphs, any vertex set can form an edge; graphs are 2-uniform hypergraphs.

Definition 6.5. *A* k-uniform hypergraph *is a hypergraph in which every edge has size k. The* transversal number $\tau(H)$ *of a hypergraph H is the minimum size of a set of vertices that intersects every edge.*

For any graph, the total domination number equals the transversal number of the hypergraph on the same vertex set in which the edges are the vertex neighborhoods. An SD-set corresponds to a transversal when the edge of the hypergraph corresponding to a vertex v of non-maximum degree is its closed neighborhood (the neighborhood plus v itself). The theorem of Chvátal and McDiarmid on transversal number of k-uniform hypergraphs provides exactly what we need to find a sufficiently small SD-set in the graph obtained by deleting the balloons. (In [7], the Chvátal–McDiarmid result is used to explore the total domination numbers of regular graphs, noting in particular that $\gamma_t(G) \le n/2$ follows immediately for cubic graphs.)

Theorem 6.6. (Chvátal and McDiarmid [4]) *If H is a k-uniform hypergraph with n vertices and m edges, then $\tau(H) \le \frac{\lfloor k/2 \rfloor m + n}{\lfloor 3k/2 \rfloor}$.*

We state the next two results for a graph G' because we will apply them when G' is the graph obtained from a 3-regular graph G by deleting the vertices in the balloons.

Corollary 6.7. *If G' is an n-vertex graph in which every vertex has degree $2r+1$ or $2r$, then G' has an SD-set of size at most $\frac{(r+1)n}{3r+1}$.*

Proof. Form the hypergraph H with $V(H) = V(G')$ by letting the edges be the open neighborhoods of vertices with degree $2r+1$ and the closed neighborhoods of vertices with degree $2r$. Thus H is a $(2r+1)$-uniform hypergraph with n vertices and n edges. By Theorem 6.6, $\tau(H) \leq \frac{(r+1)n}{3r+1}$. Every transversal of H is an SD-set in G'. □

Using the plan we described above, Corollary 6.7 implies that $\gamma_t(G) \leq \frac{n}{2} - \frac{b(G)}{2}$ when $\Delta(G) = 3$ and no two balloons have a common neighbor. The remaining case will need special attention; here deleting the balloons leaves a vertex of degree 1.

Theorem 6.8. *If G' is a connected n-vertex graph with maximum degree at most 3, and $n > 1$, then G' has a dominating set S of size at most $n/2$ such that every vertex of degree 3 has a neighbor in S.*

Proof. When $\Delta(G') < 3$, an ordinary dominating set suffices. Always some dominating set has at most $n/2$ vertices, since the complement of a minimal dominating set is also dominating. Hence we may assume that $\Delta(G') = 3$. The case $\Delta(G') < 3$ includes the basis step for induction on n.

If $\delta(G') \geq 2$, then Corollary 6.7 provides the desired SD-set. When G' has a vertex u of degree 1, let v be the neighbor of u. Let $F = G' - \{u, v\}$. If F has no isolated vertex, then we can apply the induction hypothesis to each component of F to obtain a set with the desired properties. Let T be the union of these sets; note that $|T| \leq (n-2)/2$.

If v has degree 2, then F is connected, and $T \cup \{v\}$ is an SD-set in G'.

Suppose that v has degree 3. If v has no neighbor of degree 1 other than u, then F has no isolated vertices. Now $T \cup \{v\}$ is an SD-set in G' if T contains a neighbor of v, while otherwise $T \cup \{u\}$ is an SD-set.

In the remaining case, v has degree 3 and has another neighbor w of degree 1. In this case, let $F = G' - \{u, w\}$, and let T be the set in F guaranteed by the induction hypothesis (F is connected, since we only deleted vertices of degree 1). If $v \in T$, then $T \cup \{u\}$ is an SD-set in G'. Otherwise, T must contain the remaining neighbor of v to dominate v, and now $T \cup \{v\}$ is an SD-set in G'. □

Theorem 6.9. *If G is a connected cubic graph with n vertices, then $\gamma_t(G) \leq \frac{n}{2} - \frac{b(G)}{2}$ (except that $\gamma_t(G) \leq n/2 - 1$ when $b(G) = 3$ and the three balloons have a common neighbor), and this is sharp for all even values of $b(G)$.*

Proof. Let G' be the graph obtained by deleting all vertices in balloons. If $G' = K_1$, then G consists of three balloons and their common neighbor. Lemma 6.2 yields a total dominating set in two of the balloons and a dominating set in the third that combine with one vertex of G' to yield $\gamma_t(G) \le n/2 - 1$.

When G' has more than one vertex, we can apply Theorem 6.8 to obtain an SD-set S in G'. For each balloon B, let v be the neck. Use Lemma 6.2 to add a set S_B of size $|V(B) - 1|/2$. If the neighbor of v in $V(G')$ is in S, then choose S_B to be a set that contains v and contains a neighbor of every vertex in $V(B) - \{v\}$. If the neighbor of v in $V(G')$ is not in S, then simply choose S_B to be a total dominating set of B. After these contributions from all balloons, the size is at most $\frac{n}{2} - \frac{b(G)}{2}$.

If equality holds in the bound, then G' must have no SD-set of size less than $|V(G')|/2$. Let G' be formed from a cycle C_t by adding a pendant edge at each vertex. An SD-set in G' must use one vertex from each set consisting of a vertex of degree 1 and its neighbor.

We construct our example G by adding two 7-vertex balloons adjacent to each vertex of degree 1 in G'. Each such balloon is the special balloon $\hat{B}$ discussed after Lemma 6.2. The number of balloons is $2t$. Recall that $\hat{B}$ has no total dominating set of size 3 that contains its neck. Therefore, if a total dominating set in G avoids some vertex u of degree 1 in G', then the balloons adjacent to u contribute at least four vertices each, and the 16-vertex "wedge" containing them, u, and the neighbor of u in G' contributes at least eight vertices. Using u still requires it to contribute seven vertices, including three from each balloon. Thus we can save only 1 for each pair of balloons, and $\gamma_t(G) = \frac{n}{2} - \frac{b(G)}{2}$. □

Corollary 4.2 and Theorem 6.9 together improve the inequality $\gamma_t(G) \le \alpha'(G)$ for connected cubic graphs.

Corollary 6.10. *If G is a connected n-vertex cubic graph, then $\gamma_t(G) \le \alpha'(G) - b(G)/6$, except when $b(G) = 3$ and there is exactly one vertex outside the balloons, in which case still $\gamma_t(G) \le \alpha'(G)$.*

Proof. From the bounds in Corollary 4.2 and Theorem 6.9, it suffices to consider the exceptional case. Here $b(G) = 3$, and $\gamma_t(G) = n/2 - 1$ is possible. By Lemma 6.3, there are matchings in the balloon that cover all but the neck. One of the necks can be matched to their common neighbor, leaving only the two other necks as uncovered vertices. Hence $\alpha'(G) = n/2 - 1$ (equality holds, because deleting the vertex outside the balloons leaves three odd components). □

The 3-regular case is the only case where the inequality between γ_t and α' is delicate. When more edges are added, α' tends to increase and γ_t tends to increase, so the separation increases. For $(2r + 1)$-regular graphs, applying the Chvátal–McDiarmid Theorem to the neighborhood hypergraph immediately yields $\gamma_t(G) \le \frac{(r+1)n}{3r+1}$. On the other hand, $\alpha'(G) \ge \left\lfloor \frac{2}{n}\left(1 - \frac{2r-1}{2r+1}\frac{r}{2r^2+2r-1}\right)\right\rfloor$ (Theorem 4.3). For large r, this upper bound on $\gamma_t(G)$ tends to $n/3$ and the lower bound on $\alpha'(G)$ tends to $n/2$. Already when $r = 2$, we have $\gamma_t(G) \le 3n/7 < 9n/22 < \alpha'(G)$. Hence the separation between γ_t and α' is already in the coefficient of the linear term, regardless of the number of balloons, and the balloons become important only for the 3-regular case.

Furthermore, the upper bound from the Chvátal–McDiarmid Theorem is not sharp for larger degree. The best-possible upper bounds on $\gamma_t(G)$ when G is k-regular and has n vertices are not known. Yeo [13] conjectured that if G is a connected n-vertex graph with $\delta(G) \ge 4$ other than the bipartite complement of the Heawood graph, then $\gamma_t(G) \le \frac{2}{5}n$.

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