The Chinese Postman Problem in Regular Graphs of Odd Degree

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July 22, 2011

Abstract

The Chinese Postman Problem in a graph is the problem of finding a shortest closed walk traversing all the edges. In a \((2r + 1)\)-regular graph, the problem is equivalent to finding a smallest spanning subgraph in which all vertices have odd degree. We establish a sharp upper bound for the solution in 3-regular graphs, characterize when equality holds, and conjecture the answer for general \(r\).

1 Introduction

The Chinese Postman Problem was introduced in the early 1960s by the Chinese mathematician Guan Meigu. Roughly speaking, a postman wishes to traverse every road in a city to deliver the mail, using the least possible total distance. A postman tour in a connected graph \(G\) is a closed walk containing all the edges of \(G\). An optimal postman tour in a connected graph \(G\) is a shortest closed walk traversing all edges in \(G\). Since all edges of \(G\) must be used, we are interested only in the additional length needed. Let \(p(G) = l - |E(G)|\), where \(l\) is the minimum length of a postman tour.

Since a postman tour is an Eulerian supergraph obtained by repeating some edges, \(p(G)\) equals the minimum number of edges in a parity subgraph of \(G\), where a parity subgraph is a spanning subgraph \(H\) of \(G\) such that \(d_G(v) \equiv d_H(v) \mod 2\) for every vertex \(v\) in \(G\). Thus we call \(p(G)\) the parity number of \(G\). Kostochka and Tulai [5] obtained upper bounds on

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the parity number of \((2r + 1)\)-regular 2\(t\)-edge-connected multigraphs; their bounds are sharp among multigraphs and when \(t\) is near \(r\), but they are not sharp among graphs.

We obtain a sharp upper bound on the parity number of a connected 3-regular graph with \(n\) vertices. We will show that for \(n > 10\), equality holds only for graphs in a family \(\mathcal{H}_r\) we introduced in [6] and describe in Section 2; these graphs exist when \(n\) lies in an appropriate congruence class. We conjecture that the result extends for regular graphs with larger odd degree. Let \(\mathcal{F}_{n,r}\) be the family of all connected \((2r + 1)\)-regular graphs with \(n\) vertices.

**Conjecture 1.1.** If \(G \in \mathcal{F}_{n,r}\), then \(p(G) \leq \frac{(2r^2 + 3r - 1)n - 2(r + 1)}{4r^2 + 4r - 2} - 1\). Furthermore, for \(n > 4r + 6\), equality holds if and only if \(n \equiv (4r + 6) \mod (4r^2 + 4r - 2)\) and \(G \in \mathcal{H}_r\).

In Section 3, we prove the conjecture for \(r = 1\).

## 2 The Construction

We construct a family \(\mathcal{H}_r\) of \((2r + 1)\)-regular graphs with large parity number. We introduced this family in [6], where we showed that these graphs have the most cut-edges among graphs in \(\mathcal{F}_{n,r}\) when \(n \equiv (4r + 6) \mod (4r^2 + 4r - 2)\). In addition, we showed that the graphs in \(\mathcal{F}_{n,r}\) minimizing the matching number are a particular subfamily of \(\mathcal{H}_r\) (for such \(n\)); this is generalized in [7].

**Construction 2.1.** Let \(B_r\) be the graph obtained from the complete graph \(K_{2r+3}\) by deleting a matching of size \(r + 1\) and deleting one more edge incident to the remaining vertex. This is the smallest graph in which one vertex has degree \(2r\) and the others have degree \((2r + 1)\). Note that deleting the vertex of degree \(2r\) from \(B_r\) leaves a subgraph having a perfect matching.

Let \(\mathcal{T}_r\) be the family of trees such that every non-leaf vertex has degree \(2r + 1\). Let \(\mathcal{H}_r\) be the family of \((2r + 1)\)-regular graphs obtained from trees in \(\mathcal{T}_r\) by identifying each leaf of such a tree with the vertex of degree \(2r\) in a copy of \(B_r\).

The smallest graph in \(\mathcal{H}_r\) has \(4r + 6\) vertices. Larger graphs in \(\mathcal{H}_r\) are grown from smaller ones by turning a copy of \(B_r\) back into a leaf \(x\), attaching \(2r\) new leaf neighbors to \(x\), and attaching \(B_r\) at each new leaf. The new graph has \((2r - 1)(2r + 3) + 1\) more vertices. Hence the number of vertices of each graph in \(\mathcal{H}_r\) is congruent to \(4r + 6\) modulo \(4r^2 + 4r - 2\).

We next compute the parity number for \(n\)-vertex graphs in \(\mathcal{H}_r\); this proves sharpness of Conjecture 1.1. A *balloon* in a graph \(G\) is a maximal 2-edge-connected subgraph that is incident to exactly one cut-edge of \(G\). The copies of \(B_r\) in a graph in \(\mathcal{H}_r\) are balloons, and they correspond to leaves in a tree in \(\mathcal{T}_r\). Note that \(B_r\) is the smallest possible balloon in a \((2r + 1)\)-regular graph (it plays a crucial role also in [2]). We have observed in Construction 2.1 that when growing graphs in \(\mathcal{H}_r\), each increase of \(4r^2 + 4r - 2\) in \(n\) increases the number of
balloons (leaves of the tree) by $2r - 1$ and increases the number of cut-edges (edges of the tree) by $2r$. This inductively proves the computation below.

**Proposition 2.2.** ([6]) Let $q_r = 2r^2 + 2r - 1$, and let $b(G)$ and $c(G)$ denote the numbers of balloons and cut-edges in a graph $G$, respectively. For any $n$-vertex graph $G$ in $\mathcal{H}_r$,

$$b(G) = \frac{(2r - 1)n + 2}{2q_r} \quad \text{and} \quad c(G) = \frac{r(n - 2) - 2}{q_r} - 1.$$

**Lemma 2.3.** If $G$ is regular of odd degree, then every cut-edge is in every parity subgraph.

**Proof.** Let $e$ be a cut-edge in $G$. By the Degree-Sum Formula, each component of $G - e$ has an odd number of vertices. Since a parity subgraph has odd degree at each vertex, the Degree-Sum Formula then implies that the parity subgraph must contain $e$.

Next, we determine the parity number of graphs in $\mathcal{H}_r$.

**Proposition 2.4.** If $G$ is a graph in $\mathcal{H}_r$, then

$$p(G) = \frac{(2r^2 + 3r - 1)n - 2(r + 1)}{4r^2 + 4r - 2} - 1,$$

which reduces to $\frac{2n - 5}{3}$ for 3-regular graphs.

**Proof.** Let $T$ be the tree obtained by shrinking all the balloons in $G$. Each edge of $T$ is a cut-edge in $G$ and belongs to every parity subgraph of $G$, by Lemma 2.3. Each vertex of each balloon other than the one having degree $2r$ in the balloon must also be incident to an edge in the parity subgraph, so a parity subgraph must have at least $b(G)(2r + 2)/2$ edges besides the cut-edges. Thus

$$p(G) \geq c(G) + (r + 1)b(G) = \frac{r(n - 2) - 2}{q_r} - 1 + (r + 1)\frac{(2r - 1)n + 2}{2q_r}$$

$$= \frac{(2r^2 + 3r - 1)n - 2(r + 1)}{4r^2 + 4r - 2} - 1,$$

by Proposition 2.2. By taking all edges of $T$ plus a near-perfect matching in each copy of $B_r$, equality is achieved.

# 3 The Upper Bound

**Definition 3.1.** A $k$-graph\(^1\) is a $k$-regular multigraph $G$ with an even number of vertices such that for every odd-sized subset $X$ of $V(G)$, the number of edges with exactly one endpoint in $X$ is at least $k$.

\(^1\)There are at least three different meanings for this term in the literature. Seymour [8] used the definition above, conjecturing that if $G$ is a $k$-graph, then $\chi'(G) \leq k + 1$. In Berge’s book [1], a $k$-graph is a directed multigraph with edge-multiplicity at most $k$. In [4] and elsewhere, a $k$-graph is a $k$-uniform hypergraph.
Remark 3.2. Every 2-edge-connected 3-regular multigraph is a 3-graph, since the Degree-Sum Formula requires three edges leaving every odd-sized subset $S$ of $V(G)$. More generally, if $G$ is a $(k-1)$-edge-connected $k$-regular multigraph with even order, then $G$ is a $k$-graph for the same reason. Also, every $k$-edge-colorable $k$-regular graph is a $k$-graph.

We need a fundamental result about $k$-graphs due to Edmonds.

Theorem 3.3. (Edmonds [3]) If $G$ is a $k$-graph, then there is an integer $p$ and a family $\mathcal{M}$ of perfect matchings such that each edge of $G$ is contained in precisely $p$ members of $\mathcal{M}$. (The members of $\mathcal{M}$ need not be distinct.)

Lemma 3.4. If $G$ is a $2r$-edge-connected $(2r+1)$-regular multigraph, in which each edge $e$ has weight $w(e)$, then there exists a perfect matching with weight at most $\frac{1}{2r+1}W$, where $W = \sum_{e \in E(G)} w(e)$. For 3-regular graphs, the bound reduces to $\frac{1}{3}W$.

Proof. Let $\mathcal{M}$ be a family of perfect matchings as guaranteed by Theorem 3.3. By counting two ways, $|\mathcal{M}| \frac{n}{2} = \frac{(2r+1)n}{2}p$, which yields $|\mathcal{M}| = p(2r+1)$. Let $\mathcal{M} = \{M_1, \ldots, M_{p(2r+1)}\}$, and let $w(M_i)$ be the total weight of all edges in $M_i$. Since $\sum w(M_i) = p\sum_{e \in E(G)} w(e) = pw$, a lightest matching in the family has weight at most $\frac{1}{2r+1}W$, by the pigeonhole principle. □

For the proof of the main result, we need the concept of “threads”. A thread in a graph $G$ is a maximal path in $G$ whose internal vertices have degree 2 in $G$. We also need a bound on the number of balloons in a graph in $\mathcal{F}_{n, r}$. When $r = 1$, the bound reduces to $(n+2)/6$.

Lemma 3.5. ([6]) If $G \in \mathcal{F}_{n, r}$, then $b(G) \leq \frac{(2r-1)n+2}{4r^2+4r-2}$, with equality if and only if $G \in \mathcal{H}_1'$.

Theorem 3.6. If $G \in \mathcal{F}_{n, 1}$ and $n \geq 10$, then $p(G) \leq \frac{2n-5}{3}$. Equality holds when $G \in \mathcal{H}_1'$.

Proof. Consider $G \in \mathcal{F}_{n, 1}$. If $G$ has no balloons or if $n = 10$, then $G$ has a perfect matching and $p(G) = \frac{n}{2} \leq \frac{2n-5}{3}$. Otherwise, $G$ has a balloon and $n > 10$. For $n > 10$, we proceed by induction. Let $e$ be a cut-edge. Let $G_1$ and $G_2$ be the components of $G-e$. Since a cut-edge must appear in every parity subgraph (Lemma 2.3), $p(G) = p(G_1) + p(G_2) + 1$.

Let $G_1'$ and $G_2'$ be the graphs obtained from $G$ by replacing $G_2$ and $G_1$, respectively, with $B_1$. Every parity subgraph of $G_1'$ contains $e$ and a parity subgraph of $G_i$, and it uses at least two edges in its copy of $B_1$. Thus $p(G_1') = p(G_1) + 3$, so $p(G) = p(G_1') + p(G_2') - 5$.

If neither $G_1$ nor $G_2$ is $B_1$, then $G_1'$ and $G_2'$ are smaller than $G$. Letting $n_i = |V(G_i')|$, we have $n = n_1 + n_2 - 10$. By applying the induction hypothesis to both $G_1'$ and $G_2'$,

$$p(G) = p(G_1') + p(G_2') - 5 \leq \frac{2n_1-5}{3} + \frac{2n_2-5}{3} - 5 = \frac{2n-5}{3}. \quad (1)$$

In the remaining case, every cut-edge is incident to a copy of $B_1$. Let each edge have weight 1. Form $G'$ by deleting all the vertices of all the balloons (for each balloon, we lose
eight edges). If $G'$ is a cycle, then $G$ has a perfect matching (each balloon plus its pendant edge has a perfect matching), and

$$p(G) = \frac{n}{2} < \frac{2n - 5}{3}. \quad (2)$$

Otherwise, replace each thread of $G'$ through vertices of degree 2 with a single edge whose weight is the length of the thread. Since the vertices of degree 2 have been suppressed and $G'$ is 2-edge-connected, the resulting weighted graph $G''$ is a 3-graph, by Remark 3.2. Applying Lemma 3.4, $G''$ has a perfect matching with at most $1/3$ of its total weight. The total weight is $m - 8b$, where $m$ is the number of edges in $G$ and $b$ is the number of balloons in $G$. Using Lemma 3.5 with $r = 1$, we have

$$p(G) \leq p(G') + 3b \leq \frac{m - 8b}{3} + 3b = \frac{3n - 16b}{6} + 3b = \frac{n + 3b}{2} \leq \frac{n}{2} + \frac{1}{3} \left( \frac{n + 2}{6} \right) \leq \frac{2n - 5}{3}. \quad (3)$$

We have proved that $p(G) \leq \frac{2n - 5}{3}$ for a connected 3-regular graph $G$.

By Proposition 2.4, equality holds for graphs in $\mathcal{H}_1$. \hfill \square

Since $\frac{10}{2} = \frac{2\times 10 - 5}{3}$, 10-vertex connected 3-regular graphs without cut-edges also achieve equality even though they are not in $\mathcal{H}_1$. However, a graph $G$ with more than 10 vertices satisfying $p(G) = \frac{2n - 5}{3}$ must be in $\mathcal{H}_1$.

**Theorem 3.7.** If $G \in \mathcal{F}_{n,1}$, then $p(G) = \frac{2n - 5}{3}$ if and only if $n = 10$ or $G \in \mathcal{H}_1$.

**Proof.** It suffices to show that if $G \in \mathcal{F}_{n,1}$ and $p(G) = \frac{2n - 5}{3}$, then $n = 10$ or $G \in \mathcal{H}_1$. If $n < 10$, then $G$ has a perfect matching, and $p(G) = \frac{n}{2} > \frac{2n - 5}{3}$.

For $n > 10$, we use induction on $n$ as in the proof of Theorem 3.6. To achieve equality in (1), we must have $p(G_i) = \frac{2n_i - 5}{3}$ for $i \in \{1, 2\}$. Since neither component obtained by deleting the cut-edge is $B_1$, we have $|V(G_i)| > 10$. Thus, the induction hypothesis applies, and $G'_i$ is in $\mathcal{H}_1$. Since shrinking the balloons in $G'_1$ or $G'_2$ yields a tree whose internal vertices have degree 3, the same holds also for $G$, so also $G$ is in $\mathcal{H}_1$.

In the case where all cut-edges are incident to balloons, we have three subcases. If deleting the balloons leaves a cycle, then $p(G) = \frac{n}{2} < \frac{2n - 5}{3}$. If it leaves a single vertex, then $n = 16$, $b = 3$ and $G \in \mathcal{H}_1$. If it leaves a graph with maximum degree 3, then the last part of (3) states $p(G) \leq \frac{5n + 1}{9} \leq \frac{2n - 5}{3}$, with equality only when $n = 16$. \hfill \square

**References**


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