Longest Cycles in k-connected Graphs with Given Independence Number

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University of Illinois at Urbana-Champaign

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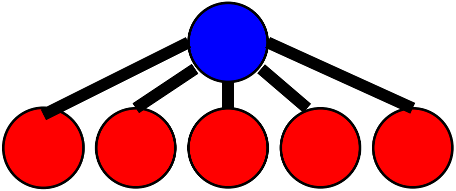
Definitions and Examples



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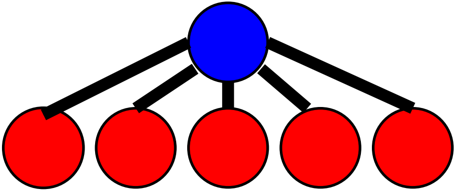

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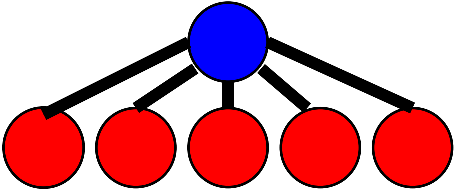

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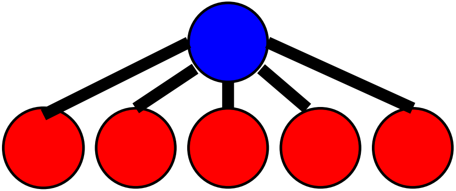

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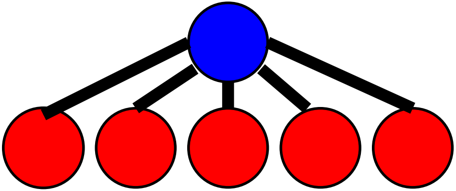

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- The join of G and H, denoted $G \vee H$, is the graph obtained from G and H by joining every vertex of G to every vertex of H.
- A graph G is said to be k-connected when there does not exist a set of $(k-1)$-vertices whose removal disconnects the graph.

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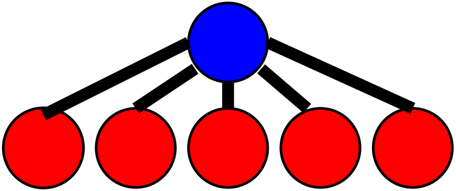

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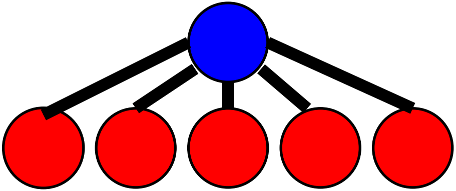

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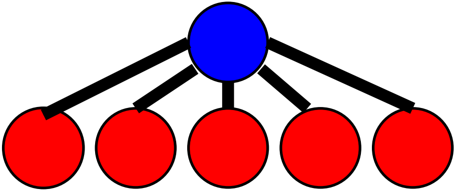

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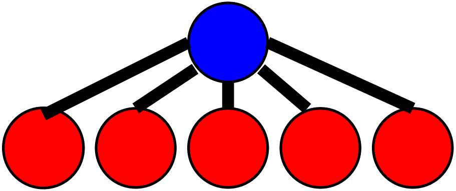

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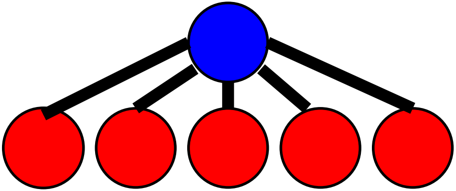

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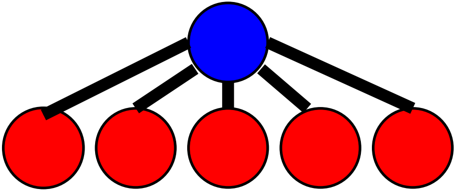

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- $\alpha(G)$: the independence number of a graph G, the maximum size of an independent set of vertices.

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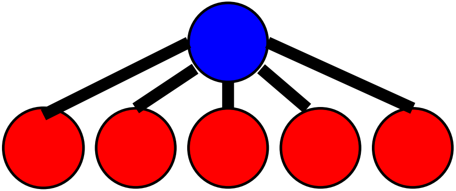

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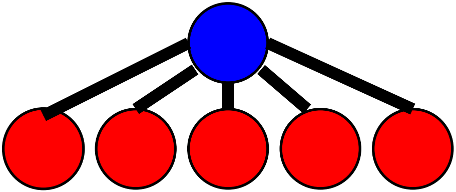

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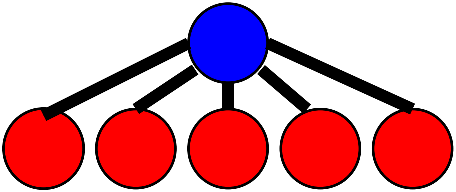

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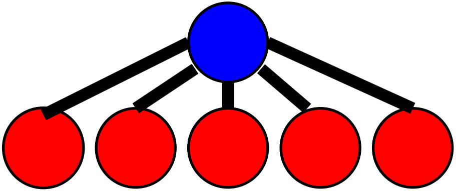

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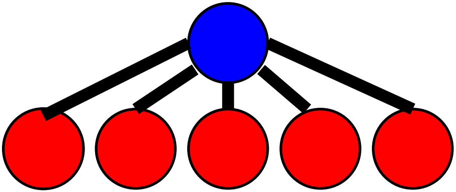

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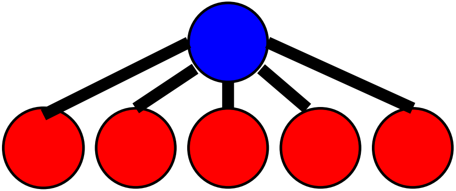

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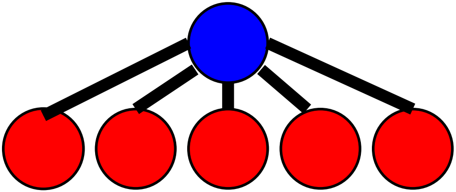

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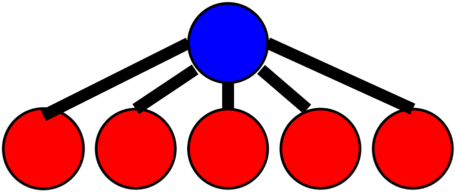

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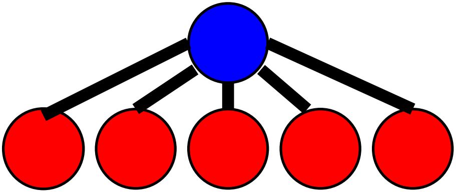

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Sufficient Conditions for Hamiltonicity

In 1952, Dirac proved that if G is a simple graph with n vertices ($n \geq 3$) and $\delta(G) \geq \frac{n}{2}$, then $c(G) = n$.

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In 1972, Chvátal and Erdős showed that if $\kappa(G) \geq \alpha(G)$ for a graph G, then $c(G) = n$.

Longest Cycle Versions for 2-connected Graphs

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Now, is there a longest cycle version of Chvátal and Erdős theorem?

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Conjecture (Fouquet-Jolivet 1976)

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$$c(G) \geq \frac{k(n + \alpha - k)}{\alpha}$$

, where $k = \kappa(G)$, and $\alpha = \alpha(G)$.

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In 2009, Manoussakis proved it for $k = 3$.

Sharpness of Fouquet-Jolivet Conjecture

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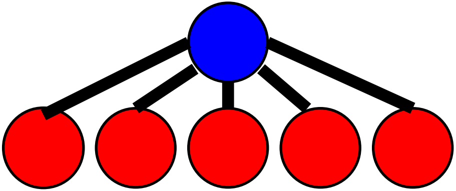

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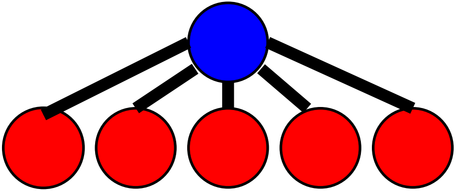

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Conjecture(Chen-Chen-Liu)

If C_1 and C_2 are distinct cycles in a k-connected graph G, then there are distinct cycles C_1' and C_2' in G such that $V(C_1) \cup V(C_2) \subseteq V(C_1') \cup V(C_2')$ and $|V(C_1') \cap V(C_2')| \geq k$.

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Without using Chen-Chen-Liu Conjecture, we prove Fouquet-Jolivet Conjecture.

The Key Lemmas

Path Lemma

If H is a subgraph of a k-connected graph G, and $u, v \in V(G)$, then G has a u, v-path P with $\alpha(H - V(P)) \leq \max\{0, \alpha(H) - (k-1)\}$.

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If H and C are disjoint subgraphs of a k-connected graph G, with C being a cycle of length $\geq k$, then G has a cycle C' such that $|V(C) - V(C')| \leq \frac{|V(C)|}{k} - 1$ and $\alpha(H - V(C')) \leq \alpha(H) - 1$.

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Multicycle Lemma

If G is a k-connected graph with $\alpha(G) = \alpha$, and $0 \leq l \leq \alpha - k$, then there exists cycles $C_0, \cdots, C_l$ satisfying the following :

(1) $\alpha(G - \cup_{i=0}^{l} V(C_i)) \leq \alpha - k - l$

(2) $|V(C_i) - \cup_{j=0}^{i-1} V(C_j)| \leq \frac{|V(C_0)|}{k} - 1$ for $1 \leq i \leq l$.

The outline of the proof of Fouquet-Jolivet Conjecture

Path Lemma ⇒ Cycle Lemma ⇒

Multicycle Lemma ⇒ Fouquet-Jolivet Conjecture

Multicycle Lemma $\Rightarrow$ Fouquet-Jolivet Conjecture

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Fouquet-Jolivet Conjecture is true.

Proof. Consider $l = \alpha - k$ in Multicle Lemma.
Thus $C_0, \cdots, C_l$ cover $V(G)$ by (1).
By (2), $n = |V(C_0)| + \sum_{i=1}^{l} |V(C_i) - \cup_{j=0}^{i-1} V(C_j)| \leq$
$|V(C_0)| + (\alpha - k)\left(\frac{|V(C_0)|}{k} - 1\right)$.
The inequality simplies to $|V(C_0)| \geq \frac{k(n+\alpha-k)}{\alpha}$.

Fouquet-Jolivet Conjecture $\Rightarrow$ Wow Theorem

Path Lemma $\Rightarrow$ Cycle Lemma $\Rightarrow$

Multicycle Lemma $\Rightarrow$ Fouquet-Jolivet Conjecture

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