On Permutations Avoiding Arithmetic Progressions

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Let S be a subset of the positive integers, and let σ be a permutation of S. We say that σ is a k**-avoiding** permutation of S if σ does not contain any k-term AP as a subsequence. Similarly, the set S is said to be k**-avoidable** if there exists a k-avoiding permutation of S.

Let $M(n)$ denote the number of 3-avoiding permutations of $\{1, 2, \ldots, n\}$. For example, $M(4) = 10$, corresponding to the permutations $(1, 2, 4, 3)$, $(1, 3, 2, 4)$, $(2, 1, 4, 3)$, $(2, 4, 1, 3)$, $(4, 2, 1, 3)$ and their reversals. In 1977, Davis, Entringer, Graham and Simmons [1] established the following bounds on $M(n)$:

$$2^{n-1} \leq M(n) \leq \lfloor (n+1)/2 \rfloor ! \lceil (n+1)/2 \rceil !$$

These bounds were recently improved by Sharma [3], who showed that

$$M(n) \leq \frac{2.7^n}{21} \text{ for } n \geq 11$$

and that

$$\lim_{n\to\infty} \frac{M(n)}{2^n\, n^k} = \infty \text{ for any fixed } k.$$

In [3] the question whether $\lim_{n\to\infty} \frac{M(n+1)}{M(n)} = 2$ was attributed to the Editor of the Problem Section of the American Mathematical Monthly (where the function $M(n)$ made its earliest known appearance, in 1975), and was mentioned as still open. We begin with an observation that settles this question in the negative. Indeed, we establish the following stronger lower bound on $M(n)$.

Theorem 1. *$M(n) \geq (1/2)c^n$ for $n \geq 8$, where $c = (2132)^{1/10} = 2.152...$.*

Proof. The following inequalities were proved in [1] to show that $M(n) \geq 2^{n-1}$:

$$M(2n) \geq 2[M(n)]^2 \; ; \; M(2n+1) \geq 2M(n)M(n+1)$$

These recurrences follow from the observation that if σ_1 and σ_2 are 3-avoiding permutations of $\{2, 4, \ldots, 2n\}$ and $\{1, 3, \ldots, 2n-1\}$ respectively, concatenating them in either order yields 3-avoiding permutations $\sigma_1\sigma_2$ and $\sigma_2\sigma_1$ of $\{1, 2, \ldots, 2n\}$, since the first and third terms of any arithmetic progression have the same parity. Note that these recurrences imply the stronger lower bound $M(n) \geq (1/2)c^n$ for $n \geq 8$, where $c = (2M(10))^{1/10} = 2.152...$. Since $M(8) = 282$, $M(9) = 496$, $M(10) = 1066$, $M(11) = 2460$, $M(12) = 6128$, $M(13) = 12840$, $M(14) = 29380$ and $M(15) = 73904$ (see [1]), the inequality holds for $8 \leq n \leq 15$. We can now use induction on k to show that it also holds for $2^k \leq n < 2^{k+1}$, $k \geq 4$. ∎

We now look at permutations of infinite subsets of the positive integers. Davis et al. [1] observed that any permutation of the positive integers contains a 3-term AP as a subsequence. (Let a_1 be the first term, and let k be the least integer such that $a_k > a_1$. Then $2a_k - a_1$ occurs to the right of both a_1 and a_k.) They also constructed a 5-avoiding permutation of the positive integers. The 4-avoidability of the positive integers remains a fascinating open problem. However, if we restrict our attention to arithmetic progressions with odd common difference, the question can be answered.

Theorem 2. *Any permutation of the positive integers must contain a 3-term AP with odd common difference as a subsequence. Furthermore, there exists a permutation of the positive integers in which no 4-term AP with odd common difference occurs as a subsequence.*

Proof. We first show that any permutation $\sigma = (t_1, t_2, \ldots, t_{11})$ of $\{1, 2, \ldots, 11\}$ with $t_1 = 2$ and $t_2 = 1$ must contain a 3-term AP with odd common difference as a subsequence. Indeed, 4 must appear in σ before 3 to avoid the 3-term AP $(2, 3, 4)$; similar considerations force 4 before 5, 7 before 4, 6 before 5, 6 before 7, 11 before 6, 8 before 11, 8 before 9 and 7 before 10. Thus, both 8 and 11 appear in σ before either 9 or 10 appears. Now we have the 3-term AP $(8, 9, 10)$ if 9 appears before 10 in σ and the 3-term AP $(11, 10, 9)$ otherwise. This proves our claim.

Let $a_1, a_2, \ldots$ be a permutation of the positive integers. We may assume without loss of generality (subtracting $a_1 - 1$ from each term and ignoring non-positive terms) that $a_1 = 1$. Let k be the least index such that a_k is even, and let $a_j = \max(a_1, a_2, \ldots, a_{k-1})$. If $a_j < 2a_k - 1$, then we have $(1, a_k, 2a_k - 1)$ as a subsequence. If $a_j \geq 2a_k - 1 > a_k$, then let $d = a_j - a_k$. Note that d is odd. Since a_j occurs before $a_k = a_j - d$ and they both occur before any of $a_j + d, a_j + 2d, \ldots, a_j + 9d$, it follows from the above claim (via shifting and scaling) that the permutation contains a 3-term AP with odd common difference.

We now exhibit a permutation of the positive integers that contains no 4-term AP with odd common difference as a subsequence. For $i \geq 1$, let σ_i be a 3-avoiding permutation of the following set of 2^{2i-1} consecutive even numbers:

$$\{(4^i + 2)/3, (4^i + 8)/3, \ldots, (4^{i+1} - 4)/3\}$$

Similarly, let π_i be a 3-avoiding permutation of the following set of 4^{i-1} consecutive odd numbers:

$$\{(4^i + 2)/6, (4^i + 14)/6, \ldots, (4^{i+1} - 10)/6\}$$

Observe that the concatenated sequence $\sigma_1\pi_1\sigma_2\pi_2\sigma_3\pi_3\cdots$ is a permutation of the positive integers. By virtue of our construction, if an odd number x occurs in this sequence before an even number y, then $2x - y < 0$. It follows that no 4-term AP with odd common difference occurs as a subsequence. ∎

Given a subset S of the positive integers, let $\overline{d}(S)$ and $\underline{d}(S)$ denote, respectively, the upper and lower densities of S. In other words,

$$\overline{d}(S) = \limsup_{n\to\infty} \frac{S(n)}{n} \quad \text{and} \quad \underline{d}(S) = \liminf_{n\to\infty} \frac{S(n)}{n} \quad \text{where } S(n) = |S \cap [1, n]|.$$

Define, for $k \geq 3$,

$$\alpha(k) = \sup_S \{\overline{d}(S) : S \text{ is } k\text{-avoidable}\} \quad \text{and} \quad \beta(k) = \sup_S \{\underline{d}(S) : S \text{ is } k\text{-avoidable}\}.$$

Since the set of positive integers is 5-avoidable, $\alpha(k) = \beta(k) = 1$ for $k \geq 5$. Bounds for $\alpha(3)$ and $\beta(3)$ were sought in [1]. We show the following:

Theorem 3. $\alpha(4) = 1$, $\alpha(3) \geq 1/2$, $\beta(4) \geq 1/3$, $\beta(3) \geq 1/4$.

Proof. For integers $a \geq 2$ and $i \geq 0$, define $S_i^{(a)} = \{a^{2i}, a^{2i} + 1, \ldots, a^{2i+1}\}$, and let σ_i^a be a 3-avoiding permutation of $S_i^{(a)}$. Define $S^{(a)} = \bigcup_{i\geq 0} S_i^{(a)}$. We claim that $S^{(a)}$ is 4-avoidable. Clearly, the concatenated sequence $\sigma_0^a\sigma_1^a\cdots$ does not contain a decreasing 3-term AP. Suppose it contains an increasing 4-term AP x_1, x_2, x_3, x_4. Since x_2, x_3 and x_4 cannot all belong to the same set $S_i^{(a)}$, we must have $x_4 \geq 2x_3$ or $x_3 \geq 2x_2$. But then $x_2 \leq 0$ or $x_1 \leq 0$, yielding a contradiction. Note that

$$\overline{d}(S^{(a)}) = \frac{a-1}{a}\sum_{i=0}^{\infty} a^{-2i} = \frac{a}{a+1} \quad \text{and} \quad \underline{d}(S^{(a)}) = \frac{a-1}{a^2}\sum_{i=0}^{\infty} a^{-2i} = \frac{1}{a+1}$$

Since a can be arbitrarily large, it follows that $\alpha(4) = 1$. Taking $a = 2$, we get $\beta(4) \geq 1/3$.

Let $p_0 = 1, q_0 = 2$, and for $k \geq 1$, define $p_k = 2q_{k-1}$ and $q_k = 3q_{k-1} - 1$. Let τ_k be a 3-avoiding permutation of $T_k = \{p_k, p_k + 1, \ldots, q_k\}$, and let $T = \bigcup_{k \geq 0} T_k$. Since $p_k = 3^k + 1 = 2q_{k-1}$ for $k \geq 1$, it follows that $\overline{d}(T) = 1/2$ and $\underline{d}(T) = 1/4$. We claim that the concatenated sequence $\tau_0\tau_1 \cdots$ is a 3-avoiding permutation of T. Indeed, if the (increasing) 3-term AP x_1, x_2, x_3 occurs as a subsequence, with x_2 and x_3 belonging to different sets T_k and T_ℓ, then $x_3 \geq 2x_2$, so $x_1 \leq 0$, yielding a contradiction. If x_2 and x_3 belong to the same set T_k, then $x_1 \in T_\ell$ with $\ell < k$. But $x_3 - x_2 < q_{k-1} \leq x_2 - x_1$, contradicting our assumption that x_1, x_2, x_3 is a 3-term AP. Therefore, T is 3-avoidable. Thus $\alpha(3) \geq 1/2$ and $\beta(3) \geq 1/4$. ∎

Erdős and Graham [2] (see also [1]) asked if the positive integers can be partitioned into two 3-avoidable sets. Clearly, the answer is negative if $\alpha(3) + \beta(3) < 1$. We believe this to be the case, and conjecture that the lower bounds in the above theorem are optimal, i.e., $\alpha(3) = 1/2$ and $\beta(3) = 1/4$. However, we have not even been able to show that $\beta(3) < 1$.

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References

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