

## UIUC Putnam Training Sessions: Binomial identities and combinatorial problems

### Tools: Binomial identities

1. **Binomial theorem.**  $\sum_{k=0}^n \binom{n}{k} x^k = (1+x)^n$  ( $n = 1, 2, \dots, x$  real)
2. **Binomial series.**  $\sum_{k=0}^{\infty} \binom{\alpha}{k} x^k = (1+x)^\alpha$  ( $|x| < 1, \alpha$  any real number),  
where  $\binom{\alpha}{k} = (\alpha)(\alpha-1)\cdots(\alpha-k+1)/k!$ .

### Problem Set 1: Binomial identities

1.  $\sum_{k=0}^n \binom{n}{k}$
2.  $\sum_{k=0}^n (-1)^k \binom{n}{k}$
3.  $\sum_{k=0}^{2n} (-1)^k k^n \binom{2n}{k}$
4.  $\sum_{k=0}^n \binom{n}{k}^2$
5.  $\sum_{k=0}^n \frac{1}{k+1} \binom{n}{k}$
6.  $\sum_{k=0}^r \binom{m}{k} \binom{n}{r-k}$
7.  $\sum_{m=0}^n \binom{m}{k}$
8.  $\sum_{k=0}^n \frac{\binom{m}{k}}{\binom{n}{k}} (n \geq m)$

### Problem Set 2: Combinatorial problems

1. How many subsets are there in a set with  $n$  elements?
2. How many of these subsets have an *even* number of elements?
3. In how many ways can 16 players be paired for the first round of a tennis tournament?
4. How many ways are there to place an order of  $n$  donuts if there are  $k$  varieties to choose from?
5. How many 10 letter “words” can be formed using 3 A’s, 2 E’s, 2 I’s, one B, one C, and one D?
6. How many ordered triples of sets (A,B,C) satisfy  $A \cap B \cap C = \emptyset$  and  $A \cup B \cup C = \{1, 2, \dots, 10\}$ ? (Putnam '85, A1)