UIUC Putnam Training Sessions: Binomial identities and combinatorial problems Hints and Solutions

Problem Set 1: Binomial identities Hints and Solutions

1. $\sum_{k=0}^{n} \binom{n}{k}$

   Answer: 2^n (Interpret the sum as the number of subsets of an n-element set).

2. $\sum_{k=0}^{n}(-1)^k \binom{n}{k}$

   Answer: 1 if $n = 0$ and 0 if $n \geq 1$ (apply the binomial theorem with $x = -1$).

3. $\sum_{k=0}^{2n}(-1)^k k^n \binom{2n}{k}$

   Answer: 0 if $n \geq 1$. (This is a seemingly impossibly hard problem that, surprisingly, becomes doable in a more general form, namely with k^n replaced by a general power k^r, $r = 0, 1, \ldots$. The case $r = 0$ is just the sum in the previous problem; to obtain the general case, one can use induction on r and Pascal's identity. The sum turns out to be 0 for $n \geq 1$.)

4. $\sum_{k=0}^{n} \binom{n}{k}^2$

   Answer: $\binom{2n}{n}$ (Write the second binomial as $\binom{n}{n-k}$ and apply Vandermonde's identity (see below)).

5. $\sum_{k=0}^{n} \frac{1}{k+1}\binom{n}{k}$

   Answer: $\frac{1}{n+1}(2^{n+1}-1)$ (Use the identity $\frac{1}{k+1}\binom{n}{k} = \frac{1}{n+1}\binom{n+1}{k+1}$. Alternatively, one could prove this using generating functions, starting from the identity $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$ and integrating from 0 to 1.)

6. $\sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k}$

   Answer: $\binom{m+n}{r}$. (The **Vandermonde identity**. For a combinatorial proof, count the number of ways to form a committee of r people from a group of m men and n women.)

7. $\sum_{m=0}^{n} \binom{m}{k}$

   Answer: $\binom{n+1}{k+1}$ (Apply Pascal's identity repeatedly. For a combinatorial proof, consider all $(k+1)$-element subsets of the set $\{1, 2, \ldots, n+1\}$, and group them according to their greatest element.)

8. $\sum_{k=0}^{n} \frac{\binom{m}{k}}{\binom{n}{k}} (n \geq m)$

   Answer: $\dfrac{n+1}{n-m+1}$. (This one is harder, as it involves quotients instead of products. The trick is to use (and prove) the identity $\binom{m}{k}/\binom{n}{k} = \binom{n-k}{n-m}/\binom{n}{m}$ to reduce the sum to one of the same type as in the previous problem.)

Problem Set 2: Combinatorial problems
Hints and Solutions

1. How many subsets are there in a set with n elements?

   Answer: 2^n (See the first problem in Set 1)

2. How many of these subsets have an *even* number of elements?

   Answer: 2^{n-1} if $n \geq 1$ (Observe that if S a subset with an even number of elements, the membership of the first $n-1$ elements in S can be decided arbitrarily, but the membership of the last element in S is completely determined by the previous $n-1$ choices. Alternatively, see the second problem in Set 1, which shows that the number of subsets with an even number of elements is equal to the number of subsets with an odd number of elements.)

3. In how many ways can 16 players be paired for the first round of a tennis tournament?

   Answer: $15 \cdot 13 \cdot 11 \cdots 3 \cdot 1 = \dfrac{16!}{2^8 8!} = \dfrac{\binom{16}{2}\binom{14}{2}\cdots\binom{2}{2}}{8!}$ (The first player can be paired with any of the 15 remaining players. Now we are left with a smaller problem of pairing 14 players, so we repeat the process or appeal to induction.)

4. How many ways are there to place an order of n donuts if there are k varieties to choose from?

   Answer: $\binom{n+k-1}{k-1}$ (Imagine the donuts lined up with $k-1$ dividers between the different varieties, for a total of $n+k-1$ spots: $k-1$ for the dividers, and n for the donuts. Then count the number of ways to pick the $k-1$ spots for the dividers out of the $n+k-1$ available spots.)

5. How many 10 letter "words" can be formed using 3 A's, 2 E's, 2 I's, one B, one C, and one D?

   Answer: $\binom{10}{3}\binom{7}{2}\binom{5}{2}\binom{3}{1}\binom{2}{1}\binom{1}{1}$ (First pick three out of the 10 available slots for the letters and place the A's in those slots, then pick two out of the remaining 7 free slots for the E's, and so on.)

6. How many ordered triples of sets (A,B,C) satisfy $A \cap B \cap C = \emptyset$ and $A \cup B \cup C = \{1, 2, \ldots, 10\}$? (Putnam '85, A1)

   Answer: Consider the Venn diagram formed by A, B and C. Each element can, independently of the others, go into six of the eight regions in the diagram. Thus there are 6^{10} triples (A, B, C) with the specified properties.