Putnam Training Session 2

Tools: Series identities

1. **Geometric series.** $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad (|x| < 1)$
2. **Finite geometric series.** $\sum_{k=0}^{n} x^k = \frac{1-x^{n+1}}{1-x} \quad (n = 1, 2, \dots, x \neq 1)$
3. **Exponential series.** $\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$
4. **Logarithmic series.** $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n} = \ln(1+x) \quad (-1 < x \leq 1)$
5. **Binomial theorem.** $\sum_{k=0}^{n} \binom{n}{k} x^k = (1+x)^n \quad (n = 1, 2, \dots, x \text{ real })$
6. **Binomial series.** $\sum_{k=0}^{\infty} \binom{\alpha}{k} x^k = (1+x)^\alpha \quad (|x| < 1, \alpha \text{ any real number})$, where $\binom{\alpha}{k} = (\alpha)(\alpha-1)\cdots(\alpha-k+1)/k!$.

Problem Set 3: Infinite series

1. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$.

   Answer: $-\ln 2$. (Put $x = 1$ in Identity 4)

2. $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.

   Answer: 1. $\left(\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}\right)$

3. $\sum_{n=k}^{\infty} x^n \quad (k = 0, 1, 2, \dots, |x| < 1)$

   Answer: $\frac{x^k}{1-x}$. (Multiply both sides of Identity 1 by x^k).

4. $\sum_{n=0}^{\infty} \frac{n}{2^n}$

   Answer: $\sum_{n=0}^{\infty} \frac{n}{2^n} = \sum_{k=1}^{\infty} \sum_{n=k}^{\infty} 2^{-n} = \sum_{k=1}^{\infty} 2^{1-k} = 2$

5. $\sum_{n=1}^{\infty} \frac{1}{n2^n}$

   Answer: $\ln 2$. (Put $x = -1/2$ in Identity 4.)

6. $\sum_{n=1}^{\infty} \frac{1}{n(n+3)}$

   Answer: $(1 + 1/2 + 1/3)/3 = 11/18$.
   $\frac{1}{n(n+1)} = \frac{1}{3}\left(\frac{1}{n} - \frac{1}{n+3}\right)$

7. $\sum_{n=0}^{\infty} \frac{(n+1)^2}{n!}$

   Answer: $5e$.
   Write $(n+1)^2 = n(n-1) + 3n + 1$ and use Identity 3

8. $\sum_{n=0}^{\infty} \binom{n+k}{k} x^n \quad (k = 0, 1, 2, \ldots, |x| < 1)$

   Answer: $\frac{1}{(1-x)^{k+1}}$

$$\sum_{n=0}^{\infty} \binom{n+k}{k} x^n = \sum_{n=0}^{\infty} \frac{(-k-1)(-k-2)\cdots(-k-n)}{k!}(-x)^n = (1-x)^{-k-1}$$

9. $\sum_{n=1}^{\infty} \dfrac{(-1)^{[2^n x]}}{2^n}$ $(0 < x < 1)$ ($[t]$ denotes the greatest integer $\leq t$)

   Answer: $1-2x$. (The numerator of the n^{th} term is $1-2k$, where $k \in \{0,1\}$ is the n^{th} digit in the binary expansion of x.)

10. $\sum_{n=1}^{\infty} \dfrac{s(n)}{n(n+1)}$, where $s(n)$ is the number of 1's in the binary expansion of n. (Putnam '81, B5)

    Answer: $2\ln 2$.
    Let $S = \sum_{n=1}^{\infty} \dfrac{s(n)}{n(n+1)}$. Since $s(2n) = s(n)$ and $s(2n+1) = s(2n) + 1$, it follows that

$$S - \frac{s(1)}{2} = \sum_{n=1}^{\infty} \frac{s(2n)}{2n(2n+1)} + \frac{s(2n+1)}{(2n+1)(2n+2)} = \sum_{n=1}^{\infty} \frac{s(n)}{2n(n+1)} + \frac{1}{(2n+1)(2n+2)}$$

Thus, $S - 1/2 = S/2 + 1/3 - 1/4 + \ldots$, i.e, $S = 2\ln 2$.