Putnam Training Session 3

Tools: Basic Inequalities

1. **Cauchy's Inequality**

$$\left(\sum a_i b_i\right)^2 \leq \left(\sum a_i^2\right)\left(\sum b_i^2\right)$$

$$\left(\int f(x)g(x)dx\right)^2 \leq \left(\int f(x)^2 dx\right)\left(\int g(x)^2 dx\right)$$

2. **Arithmetic-Geometric Mean Inequality:** If $a_i \geq 0$,

$$\left(\prod_{i=1}^{n} a_i\right)^{1/n} \leq \frac{1}{n}\sum_{i=1}^{n} a_i.$$

3. **Jensen's inequality:** A real-valued function $f(x)$ is called convex if $f((x_1+x_2)/2) \leq (f(x_1)+f(x_2))/2$ for all real x_1, x_2. If $f(x)$ is convex, and $p_i \geq 0$, $\sum p_i = 1$, then for any real x_i

$$f\left(\sum p_i x_i\right) \leq \sum p_i f(x_i).$$

Problem Set 4: Inequalities

1. Given n positive real numbers with sum 1, show that the sum of the squares of these numbers is at least $1/n$.

2. Given n positive real numbers $a_1, \ldots, a_n$, define

$$H = n\left(\frac{1}{a_1} + \cdots + \frac{1}{a_n}\right)^{-1}.$$

(The number H is called the **harmonic mean** of the numbers a_i.) Show that $H \leq G$, where $G = (a_1 \ldots a_n)^{1/n}$ is the geometric mean of the a_i's.

3. Let $a_1, \ldots, a_n$ be positive integers, and let $b_1, \ldots, b_n$ be a permutation of the a_i's. Show that $\sum_{i=1}^{n}(a_i/b_i) \geq n$.

4. Suppose f is a nonnegative function defined on the interval $[0,1]$ and satisfying $\int_0^1 f(x)^2 dx = 1$. What is the maximum value of $\int_0^1 f(x)x^{2002}dx$?

5. Let $x_1, \dots, x_n$ be real numbers with $0 < x_i < 1$, and let $x = (1/n) \sum_{i=1}^n x_i$ be the arithmetic mean of these numbers. Show that
$$\prod_{i=1}^n \left(\frac{\sin x_i}{x_i} \right) \le \left(\frac{\sin x}{x} \right)^n .$$

6. Let u, v, w be real numbers. Show that
$$\frac{u+v+w}{3} \le \log \frac{e^u + e^v + e^w}{3}.$$
When does equality hold?

7. Suppose $x_1, \dots, x_n$ are positive real numbers with $\sum_{i=1}^n x_i = 1$. Show that
$$\sum_{i=1}^n x_i \log x_i \le \log \sum_{i=1}^n x_i^2.$$