Putnam Training Session 3

Tools: Basic Inequalities

1. **Cauchy's Inequality**

$$\left(\sum a_i b_i\right)^2 \le \left(\sum a_i^2\right)\left(\sum b_i^2\right)$$

$$\left(\int f(x)g(x)dx\right)^2 \le \left(\int f(x)^2 dx\right)\left(\int g(x)^2 dx\right)$$

2. **Arithmetic-Geometric Mean Inequality:** If $a_i \ge 0$,

$$\left(\prod_{i=1}^n a_i\right)^{1/n} \le \frac{1}{n}\sum_{i=1}^n a_i.$$

3. **Jensen's inequality:** A real-valued function $f(x)$ is called convex if $f((x_1+x_2)/2) \le (f(x_1)+f(x_2))/2$ for all real x_1, x_2. If $f(x)$ is convex, and $p_i \ge 0$, $\sum p_i = 1$, then for any real x_i

$$f\left(\sum p_i x_i\right) \le \sum p_i f(x_i).$$

Problem Set 4: Inequalities
Hints and Solutions

1. Given n positive real numbers with sum 1, show that the sum of the squares of these numbers is at least $1/n$.

   Answer: Apply Cauchy-Schwarz to the sum $1 = \sum_{i=1}^n 1 \cdot a_i$.

2. Given n positive real numbers $a_1, \dots, a_n$, define

$$H = n\left(\frac{1}{a_1} + \cdots + \frac{1}{a_n}\right)^{-1}.$$

   (The number H is called the **harmonic mean** of the numbers a_i.) Show that $H \le G$, where $G = (a_1 \dots a_n)^{1/n}$ is the geometric mean of the a_i's.

   Answer: Note that H is the reciprocal of the arithmetic mean of the numbers $b_i = 1/a_i$, while G is the reciprocal of the geometric mean of these numbers. Then apply the Arithmetic-Geometric Mean (AGM) inequality.

3. Let $a_1, \dots, a_n$ be positive integers, and let $b_1, \dots, b_n$ be a permutation of the a_i's. Show that $\sum_{i=1}^n (a_i/b_i) \ge n$.

   Answer: Note that $\prod_{i=1}^n (a_i/b_i) = 1$. Apply AGM to the numbers a_i/b_i.

4. Suppose f is a nonnegative function defined on the interval $[0,1]$ and satisfying $\int_0^1 f(x)^2 dx = 1$. What is the maximum value of $\int_0^1 f(x)x^{2002}dx$?

   Answer: Let I denote the integral $\int_0^1 f(x)x^{2002}dx$. Apply the integral version of Cauchy-Schwarz with the functions $f(x)$ and x^{2002} to get $I^2 \leq \int_0^1 f(x)^2 dx \int_0^1 x^{4004} dx = 1/4005$, or $I \leq 1/\sqrt{4005}$. To show that this upper bound can be achieved, take f to be proportional to x^{2002}, i.e., $f(x) = cx^{2002}$, with $c = \sqrt{4005}$.

5. Let $x_1, \dots, x_n$ be real numbers with $0 < x_i < 1$, and let $x = (1/n)\sum_{i=1}^n x_i$ be the arithmetic mean of these numbers. Show that

$$\prod_{i=1}^n \left(\frac{\sin x_i}{x_i}\right) \leq \left(\frac{\sin x}{x}\right)^n.$$

   Answer: Taking logarithms the inequality to be shown is equivalent to $\sum_{i=1}^n \log((\sin x_i)/x_i) \leq n \log((\sin x)/x)$. Note that

$$f(x) = \log((\sin x)/x) = \log(\sin x) - \log x$$

   is concave on $(0,1)$, as

$$f''(x) = \frac{1}{x^2} - \frac{1}{\sin^2 x} < 0.$$

   The result now follows from Jensen's inequality.

6. Let u, v, w be real numbers. Show that

$$\frac{u+v+w}{3} \leq \log \frac{e^u + e^v + e^w}{3}.$$

   When does equality hold?

   Answer: Exponentiate both sides, then apply Jensen's inequality with the function $f(x) = e^x$ to get $f((1/3)u + (1/3)v + (1/3)w) \leq (1/3)(f(u) + f(v) + f(w))$.

7. Suppose $x_1, \dots, x_n$ are positive real numbers with $\sum_{i=1}^n x_i = 1$. Show that

$$\sum_{i=1}^n x_i \log x_i \leq \log \sum_{i=1}^n x_i^2.$$

   Answer: Apply Jensen's inequality with $f(x) = -\log x$ (which is convex since $\log x$ is concave) and $p_i = x_i$.