

## Putnam Training Session 4

### Tools: Basic Inequalities

1. **Factor Theorem:** The polynomial  $p(x) = a_n x^n + \dots + a_1 x + a_0$  has a root  $\alpha$  of multiplicity  $m$ , then  $p(x) = (x - \alpha)^m q(x)$ ,  $q(\alpha) \neq 0$ .
2. **Elementary Symmetric Polynomials:** Every symmetric polynomial in  $x_1, x_2, \dots, x_n$  can be expressed as a polynomial in  $\sigma_1, \sigma_2, \dots, \sigma_n$  where
 
$$\sigma_k = \sum_{1 \leq j_1 < j_2 < \dots < j_k \leq n} x_{j_1} x_{j_2} \cdots x_{j_k}$$
3. **Vieta's Formula:** Let  $z_1, z_2, \dots, z_n$  be the (possibly complex) roots of the monic polynomial  $p(x) = x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ . Then  $a_{n-k} = (-1)^k \sigma_k(z_1, z_2, \dots, z_n)$  where  $\sigma_k$  is the elementary symmetric polynomial of degree  $k$  in  $n$  variables.
4. **Identity Theorem:** If  $p(x)$  and  $q(x)$  are polynomials of degree at most  $n$ , and  $p(x_k) = q(x_k)$ ,  $1 \leq k \leq n+1$  for distinct  $x_1, x_2, \dots, x_{n+1}$ , then  $p(x) = q(x)$  for all  $x$ .

### Problem Set 5: Polynomials

1. Let  $\alpha = 2^{1/3} + 5^{1/2}$ . Find a polynomial  $p(x)$  with integer coefficients satisfying  $p(\alpha) = 0$ .
2. Find a polynomial of degree at most 3 such that  $p(2) = 3, p(3) = 5, p(5) = 8$  and  $p(7) = 13$ .
3. If  $x + y + z = 3, x^2 + y^2 + z^2 = 5, x^3 + y^3 + z^3 = 7$ , find  $x^4 + y^4 + z^4$ .
4. Find all polynomials  $P(x)$  satisfying  $P(x^2 + 1) = (P(x))^2 + 1$  for all  $x$  and  $P(0) = 0$ .
5. Find a non-zero polynomial  $P(x, y)$  such that  $P(\lfloor t \rfloor, \lfloor 2t \rfloor) = 0$  for all real numbers  $t$ . (Putnam '05, B1)
6. Suppose that the monic polynomial  $p(x) = x^n + a_{n-1} x^{n-1} + \dots + a_1 x + 1$  has non-negative coefficients and  $n$  real roots. Show that  $p(2) \geq 3^n$ .
7. Let  $p(x) = a_n x^n + \dots + a_1 x + a_0$  be a polynomial with integer coefficients. If  $r$  is a rational root of  $p(x)$ , show that the numbers  $a_n r, a_n r^2 + a_{n-1} r, \dots, a_n r^n + a_{n-1} r^{n-1} + \dots + a_1 r$  are all integers. (Putnam '04, B1)