Putnam Training Session 4

Tools: Basic Inequalities

1. **Factor Theorem:** The polynomial $p(x) = a_n x^n + \ldots + a_1 x + a_0$ has a root α of multiplicity m, then $p(x) = (x-\alpha)^m q(x),\ q(\alpha) \neq 0$.

2. **Elementary Symmetric Polynomials:** Every symmetric polynomial in $x_1, x_2, \ldots, x_n$ can be expressed as a polynomial in $\sigma_1, \sigma_2, \ldots \sigma_n$ where
$$\sigma_k = \sum_{1 \le j_1 < j_2 < \cdots < j_k \le n} x_{j_1} x_{j_2} \cdots x_{j_k}$$

3. **Vieta's Formula:** Let $z_1, z_2, \ldots z_n$ be the (possibly complex) roots of the monic polynomial $p(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$. Then $a_{n-k} = (-1)^k \sigma_k(z_1, z_2, \ldots, z_n)$ where σ_k is the elementary symmetric polynomial of degree k in n variables.

4. **Identity Theorem:** If $p(x)$ and $q(x)$ are polynomials of degree at most n, and $p(x_k) = q(x_k), 1 \le k \le n+1$ for distinct $x_1, x_2, \ldots, x_{n+1}$, then $p(x) = q(x)$ for all x.

Problem Set 4: Polynomials
Hints and Solutions

1. Let $\alpha = 2^{1/3} + 5^{1/2}$. Find a polynomial $p(x)$ with integer coefficients satisfying $p(\alpha) = 0$.

   Answer: $(\alpha - \sqrt{5})^3 = 2 \Rightarrow \alpha^3 + 15\alpha - 2 = \sqrt{5}(3\alpha^2 + 5)$. Therefore,
$$(\alpha^3 + 15\alpha - 2)^2 - 5(3\alpha^2 + 5)^2 = 0.$$
   Thus, $p(x) = (x^3 + 15x - 2)^2 - 5(2x^2 + 5)$ has the required property.

2. Find a polynomial of degree at most 3 such that $p(2) = 3, p(3) = 5, p(5) = 8$ and $p(7) = 13$.

   Answer: Let $p(x) = a + b(x-2) + c(x-2)(x-3) + d(x-2)(x-3)(x-5)$. Then $a = p(2) = 3; p(3) = a + b \Rightarrow b = 2; p(5) = a + 3b + 6c \Rightarrow c = -1/6; p(7) = a + 5b + 20c + 40d \Rightarrow d = -1/12$. Thus,
$$p(x) = 3 + 2(x-2) - \frac{(x-2)(x-3)}{6} - \frac{(x-2)(x-3)(x-5)}{12}$$

3. If $x + y + z = 3, x^2 + y^2 + z^2 = 5, x^3 + y^3 + z^3 = 7$, find $x^4 + y^4 + z^4$.

   Answer: Let $\sigma_1(x,y,z) = x + y + z,\ \sigma_2(x,y,z) = xy + yz + xz$ and $\sigma_3(x,y,z) = xyz$ denote the elementary symmetric polynomials in x, y and

z. We have, $\sigma_1 = 3, \sigma_1^2 - 2\sigma_2 = 5$ and $7 - \sigma_3 = \sigma_1(\sigma_1^2 - 3\sigma_2)$. Thus, $\sigma_2 = 2$ and $\sigma_3 = -2$. Now $x^4 + y^4 + z^4 = (x^2 + y^2 + z^2)^2 - 2(\sigma_2^2 - 2\sigma_1\sigma_3) = -7$. In particular, the given system has no real solutions.

4. Find all polynomials $P(x)$ satisfying $P(x^2 + 1) = (P(x))^2 + 1$ for all x and $P(0) = 0$.

   Answer: Consider the sequence $\{u_k\}$ defined as follows: $u_0 = 0; u_k = u_{k-1}^2 + 1$ for $k \geq 1$. It can be easily proved by induction on k that $P(u_k) = u_k$ for all k. Since $u_k > u_{k-1}$ for all k, $P(x)$ coincides with $Q(x) = x$ for infinitely many values. It follows from the Identity Theorem that $P(x) = x$.

5. Find a non-zero polynomial $P(x, y)$ such that $P(\lfloor t \rfloor, \lfloor 2t \rfloor) = 0$ for all real numbers t. (Putnam '05, B1)

   Answer: Let $\lfloor t \rfloor = n$. Thus $n \leq t < n + 1$, i.e, $2n \leq 2t < 2n + 2$. It follows that $\lfloor 2t \rfloor = 2\lfloor t \rfloor$ or $\lfloor 2t \rfloor = 2\lfloor t \rfloor + 1$. Thus $P(x, y) = (y - 2x)(y - 2x - 1)$ satisfies $P(\lfloor t \rfloor, \lfloor 2t \rfloor) = 0$ for all t.

6. Suppose that the monic polynomial $p(x) = x^n + a_{n-1}x^{n-1} + \ldots a_1x + 1$ has non-negative coefficients and n real roots. Show that $p(2) \geq 3^n$.

   Answer: Let $y_1, y_2, \ldots y_n$ be the roots of $p(x)$. Since $p(x) \geq 1$ when $x \geq 0$, we have $y_i < 0$ for all i. Let $y_i' = -y_i$. Note that

   $$P(2) = (2 + y_1')(2 + y_2') \cdots (2 + y_n').$$

   By Vieta's formula, $y_1'y_2' \cdots y_n' = 1$. Also, by the the AM-GM inequality, $2 + y_i' \geq 3(y_i')^{1/3}$. It follows that $P(2) \geq 3^n$.

7. Let $p(x) = a_nx^n + \ldots + a_1x + a_0$ be a polynomial with integer coefficients. If r is a rational root of $p(x)$, show that the numbers $a_nr, a_nr^2 + a_{n-1}r, \ldots, a_nr^n + a_{n-1}r^{n-1} + \ldots a_1r$ are all integers. (Putnam '04, B1)

   Answer: Let $r = b/c$, with $(b, c) = 1$ (i.e., b and c are relatively prime). Since $p(r) = 0$, we get, after clearing denominators,

   $$a_nb^n + a_{n-1}b^{n-1}c + \ldots, +a_0c^n = 0.$$

   For $1 \leq k \leq n$, define $p_k(b, c) = a_nb^n + a_{n-1}b^{n-1}c + \ldots + a_{n-k+1}b^{n-k+1}c^{k-1}$. Note that $c^k | p_k(b, c)$. But $p_k(b, c) = b^{n-k}(a_nb^k + a_{n-1}b^{k-1} + \ldots + a_{n-k+1}bc^{k-1})$. Furthermore, $(b, c) = 1 \Rightarrow (b^{n-k}, c^k) = 1$. Thus,

   $$c^k | a_nb^k + a_{n-1}b^{k-1}c + \ldots + a_{n-k+1}bc^{k-1}.$$

   It follows that $a_nr^k + a_{n-1}r^{k-1} + \ldots + a_{n-k+1}r$ is an integer for $1 \leq k \leq n$.