# Putnam Training Session 5

## The Pigeonhole Principle (or Box Principle)

If $n+1$ objects ("pigeons") are distributed among $n$ boxes ("pigeon holes"), at least one of the boxes contains more than one object. More generally, if $kn+1$ objects are distributed among $n$ boxes, at least one of the boxes contains more than $k$ objects.

## Problem Set 6: Pigeonhole problems

1. Show that among any five points inside an equilateral triangle of side length 1, there exist two points whose distance is at most 1/2.

2. Given a set of 7 integers, show that there exist two of them whose difference or sum is divisible by 10.

3. Prove that from a set of ten distinct two-digit integers it is possible to select two disjoint non-empty subsets whose members have the same sum.

4. Show that any set $A \subset \{1, 2, \ldots, 2n\}$ with at least $n+1$ elements contains two elements, one of which divides the other.

5. Let $S$ be the set of real numbers of the form $a + b\sqrt{2}$, where $a$ and $b$ are integers. Show that $S$ is *dense* on the real line, in the sense that, given any $\epsilon > 0$ and any real number $x$ there exists an element $s \in S$ with $|s - x| < \epsilon$.

6. The Fibonacci sequence is defined by $F_0 = F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Show that, given any positive integer $k$, there exists a Fibonacci number $F_n$ ending in at least $k$ zeros.

7. Suppose $\mathcal{A}$ is a collection of subsets of $\{1, 2, \ldots, n\}$ with the property that any two sets in $\mathcal{A}$ have a non-empty intersection. Show that $\mathcal{A}$ has at most $2^{n-1}$ elements. Can the bound $2^{n-1}$ be lowered?

8. A partition of a set $S$ is a collection of disjoint non-empty subsets (parts) whose union is $S$. For a partition $\pi$ of $\{1, 2, \ldots, 9\}$, let $\pi(x)$ be the number of elements in the part containing $x$. Prove that for any two partitions $\pi$ and $\pi'$, there exist $x, y \in \{1, 2, \ldots, 9\}, x \neq y$, such that $\pi(x) = \pi(y)$ and $\pi'(x) = \pi'(y)$. (Putnam '95, B1)