Putnam Training Session 5

The Pigeonhole Principle (or Box Principle)

If $n + 1$ objects ("pigeons") are distributed among n boxes ("pigeon holes"), at least one of the boxes contains more than one object. More generally, if $kn + 1$ objects are distributed among n boxes, at least one of the boxes contains more than k objects.

Problem Set 6: Pigeonhole Principle
Hints and Solutions

1. Show that among any five points inside an equilateral triangle of side length 1, there exist two points whose distance is at most 1/2.

   Answer: Divide the triangle into four congruent equilateral triangles of side length 1/2. Then use the pigeonhole principle to conclude that one of these must contain two points.

2. Given a set of 7 integers, show that there exist two of them whose difference or sum is divisible by 10.

   Answer: Split the remainders upon division by 10 into the 6 classes $\{0\}$, $\{1, 9\}$, $\{2, 8\}$, $\{3, 7\}$, $\{4, 6\}$, and $\{5\}$, and argue that if two integers fall into the same class then either their difference or their sum is congruent to 0 modulo 10.

3. Prove that from a set of ten distinct two-digit integers it is possible to select two disjoint non-empty subsets whose members have the same sum.

   Answer: The disjointness requirement is a red herring, since if two non-disjoint sets have the same sum, removing the common elements from each set leaves two disjoint sets that still have the same sum. Thus, we can ignore the disjointness requirement. The problem then becomes a relatively easy pigeonhole problem, with the possible values for the sums as pigeonholes, and the sets (of which there are $2^{10} - 1 = 1023$) as pigeons.

4. Show that any set $A \subset \{1, 2, \dots, 2n\}$ with at least $n + 1$ elements contains two elements, one of which divides the other.

   Answer: Write each element in A as $2^k a$ with a odd. There are n possible values for a, so if A has $n + 1$ elements, two of these must have the same value for a. These two have the required divisibility property.

5. Let S be the set of real numbers of the form $a + b\sqrt{2}$, where a and b are integers. Show that S is *dense* on the real line, in the sense that,

given any $\epsilon > 0$ and any real number x there exists an element $s \in S$ with $|s - x| < \epsilon$.

Answer: First, by considering pairs (a, b) with $0 \leq a, b \leq N$ for a large N, show that, for any ϵ, there exist two numbers of the given form whose difference is $\leq \epsilon$. Next, noting that the difference of two numbers of this form has again the same form, conclude that there exists a number of the required form with absolute value $\leq \epsilon$. Finally, by considering integer multiples of that number, show that given any x, there exists such a number with distance $\leq \epsilon$ of x.

6. The Fibonacci sequence is defined by $F_0 = F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Show that, given any positive integer k, there exists a Fibonacci number F_n ending in at least k zeros.

   Answer: Fix k, and consider the pairs $(a_n, a_{n-1}) = (F_n \bmod 10^k, F_{n-1} \bmod 10^k)$. Note that, by the Fibonacci recurrence, a single such pair determines the entire sequence $\{a_n\}$ uniquely (forwards and backwards!). Since there are only finitely many (namely, 10^{2k}) possible values for these pairs, the sequence (a_n, a_{n-1}) must be periodic. Since $(a_1, a_0) = (1, 1)$, there are infinitely many n with $(a_n, a_{n-1}) = (1, 1)$, and for each of these n, we have $a_{n-2} = 0$, by the Fibonacci recurrence.

7. Suppose $\mathcal{A}$ is a collection of subsets of $\{1, 2, \ldots, n\}$ with the property that any two sets in $\mathcal{A}$ have a non-empty intersection. Show that $\mathcal{A}$ has at most 2^{n-1} elements. Can the bound 2^{n-1} be lowered?

   Answer: Split the 2^n subsets into pairs of the form $\{A, A^c\}$. Note that a set $\mathcal{A}$ with the given property can contain at most one element from each such pair. Since there are 2^{n-1} such pairs, $\mathcal{A}$ can have at most 2^{n-1} elements.

8. A partition of a set S is a collection of disjoint non-empty subsets (parts) whose union is S. For a partition π of $\{1, 2, \ldots, 9\}$, let $\pi(x)$ be the number of elements in the part containing x. Prove that for any two partitions π and π', there exist $x, y \in \{1, 2, \ldots, 9\}, x \neq y$, such that $\pi(x) = \pi(y)$ and $\pi'(x) = \pi'(y)$. (Putnam '95, B1)

   Answer: Note that, for a given π, there can be at most 3 distinct values of $\pi(x)$ (since $1 + 2 + 3 + 4 = 10 > 9$). Thus, if $\pi(x) \geq 4$ for some x, two elements in the set containing x will share the same value of π', and we are done. If not, given π and π' there are at most 9 possible values for the pair $(\pi(x), \pi'(x))$, namely $\{(i, j) : 1 \leq i, j \leq 3\}$. If one of these pairs occurs for more than one x, we are done. Otherwise, each of these pairs occurs exactly once. Therefore,

   $$|\{x : \pi(x) = 1\}| = |\{x : \pi(x) = 2\}| = |\{x : \pi(x) = 3\}| = 3$$

   But $\{x : \pi(x) = 2\}$ has an even number of elements, so this is impossible.