Putnam Training Session 6

Tools: Useful Results

1. **Euler Totient Function:** The Euler totient function $\varphi(n)$, denoting the number of positive integers not exceeding n and relatively prime to n is given by
$$\varphi(n) = n \prod_{p_i | n} \left(1 - \frac{1}{p_i}\right)$$

2. **Euler's Theorem:** If a and n are relatively prime integers, then $a^{\varphi(n)} \equiv 1(\text{mod n})$ where $\varphi(n)$ is the Euler totient function.

3. **Pythagorean Triples:** All relatively prime positive integer solutions to $x^2 + y^2 = z^2$ with x odd and y even are of the form $x = u^2 - v^2, y = 2uv, z = u^2 + v^2$.

Problem Set 7: Number Theory

1. Let p_n be the n^{th} prime number. Show that the sequence $\{q_n\}$ defined by $q_n = p_{n+1} - p_n$ is unbounded.

2. Show that the product of four consecutive positive integers is never a perfect square.

3. Find all solutions to $1! + 2! + 3! \ldots + n! = m^2$ in positive integers.

4. Let $a > 1$. Show that $a^n + 1$ is prime only if a is even and $n = 2^k$.

5. Find all prime numbers of the form $n^4 + 4^n$.

6. Let T be a right triangle with integer sides. Show that the area of T is a multiple of 6.

7. Find the remainder when 2^{2009} is divided by 2009.

8. Suppose that a positive integer n is the sum of squares of two integers. Show that $2n, 5n$ and $10n$ are sums of squares of two integers.

9. Let $f(n)$ be the largest power of 5 dividing $1^1 2^2 \cdots n^n$. Find $\lim_{n\to\infty} \frac{f(n)}{n^2}$ (Putnam '81, A1)

10. Let $\{f(n)\}$ be a strictly increasing sequence of positive integers such that $f(2) = 2$ and $f(mn) = f(m)f(n)$ whenever m and n are relatively prime. Show that $f(n) = n$ for all n. (Putnam '63, B2)