Putnam Training Session 5

Tools: Useful Results

1. **Euler Totient Function:** The Euler totient function $\varphi(n)$, denoting the number of positive integers not exceeding n and relatively prime to n is given by
$$\varphi(n) = n \prod_{p_i | n} \left(1 - \frac{1}{p_i}\right)$$
2. **Euler's Theorem:** If a and n are relatively prime integers, then $a^{\varphi(n)} \equiv 1(\text{mod n})$ where $\varphi(n)$ is the Euler totient function.
3. **Pythagorean Triples:** All relatively prime positive integer solutions to $x^2 + y^2 = z^2$ with x odd and y even are of the form $x = u^2 - v^2, y = 2uv, z = u^2 + v^2$.

Problem Set 6: Number Theory
Hints and Solutions

1. Let p_n be the n^{th} prime number. Show that the sequence $\{q_n\}$ defined by $q_n = p_{n+1} - p_n$ is unbounded.

   Answer: Note that none of the numbers $n! + 2, n! + 3, \dots, n! + n$ are prime. Thus there are arbitrarily large gaps in the sequence of primes.

2. Show that the product of four consecutive positive integers is never a perfect square.

   Answer: Since $n(n+1)(n+2)(n+3) = (n^2+3n)(n^2+3n+2) = (n^2+3n+1)^2 - 1$, the product of four integers is one less than a perfect square. Since $n > 0$, the product is at least 24, but the only integer solutions to $x^2 - y^2 = 1$ are $(1, 0)$ and $(-1, 0)$. Thus the product is not a perfect square.

3. Find all solutions to $1! + 2! + 3! \ldots + n! = m^2$ in positive integers.

   Answer: Since $n!$ is a multiple of 10 for $n \geq 5$. Thus the last digit of $1! + 2! + 3! \ldots n!$ is 3 for $n \geq 5$. Since no perfect square ends in 3, it suffices to check $n \leq 4$. Thus the only solutions are $m = n = 3$ and $m = n = 1$.

4. Let $a > 1$. Show that $a^n + 1$ is prime only if a is even and $n = 2^k$.

   Answer: If a is odd, $a^n + 1$ is an even number greater than 2. If $n = mq$, where m is odd, then $a^q + 1 | a^n + 1$. (In general, $b + 1 | b^m + 1$ if m is odd.) Thus, if $a^n + 1$ is prime for $a > 1$, then a must be even, and n should have no odd factor, i.e., n is a power of 2.

5. Find all prime numbers of the form $n^4 + 4^n$.

   Answer: If n is even, $n^4 + 4^n$ is a multiple of 16, and therefore not prime. If n is odd, let $n + 1 = 2m$. Then

   $$n^4 + 4^n = (n^2 + 2^n)^2 - (2^m n)^2 = (n^2 + 2^n - 2^m n)(n^2 + 2^n + 2^m n)$$

   For both terms are at least 5, so that $n^4 + 4^n$ has non-trivial divisors. Thus the only prime number of the form $n^4 + 4^n$ is 5, corresponding to $n = 1$.

6. Let T be a right triangle with integer sides. Show that the area of T is a multiple of 6.

   Answer: Let the sides of T be $x^2 - y^2, 2xy$ and $x^2 + y^2$, so that the area equals $A(x, y) = xy(x - y)(x + y)$. If x or y is even, $A(x, y)$ is clearly even. If x and y are both odd, $x - y$ is even, so $A(x, y)$ is still even. Similarly, if x or y is a multiple of 3, then $A(x, y)$ is clearly a multiple of 3. If not, $x \equiv \pm y (\text{mod } 3)$, so that $x - y$ or $x + y$, and therefore $A(x, y)$ is a multiple of 3. It follows that $A(x, y)$ is always a multiple of 3.

7. Find the remainder when 2^{2009} is divided by 2009.

   Answer: Since $2009 = 41 \cdot 49$, and $\varphi(49) = 42$,

   $$2^{2009} \equiv 2^9 (\text{mod } 41) \equiv 20 (\text{mod } 41)$$

   and

   $$2^{2009} \equiv 2^{35} (\text{mod } 49) \equiv 30^5 (\text{mod } 49) \equiv 18 (\text{mod } 49),$$

   it follows that the remainder upon dividing 2^{2009} by 2009 is 1537.

8. Suppose that a positive integer n is the sum of squares of two integers. Show that $2n, 5n$ and $10n$ are sums of squares of two integers.

   Answer: Observe that $(a^2 + b^2)(c^2 + d^2) = (ac + bd)^2 + (ad - bc)^2$. Since $2 = 1^2 + 1^2, 5 = 2^2 + 1^2$ and $10 = 3^2 + 1^2$, it follows that $2n, 5n$ and $10n$ are sums of two squares.

9. Let $f(n)$ be the largest power of 5 dividing $1^1 2^2 \cdots n^n$. Find $\lim_{n\to\infty} \frac{f(n)}{n^2}$ (Putnam '81, A1)

   Answer: Let $m = 5^a b$ where $(b, 5) = 1$. Then m^m contributes am to $f(n)$. We can think of this as each multiple m of 5 contributing m, each multiple of 25 contributing an additional m, and so on. Thus,

   $$f(n) = 5(1 + 2 + \ldots + \lfloor n/5 \rfloor) + 5^2(1 + 2 + \ldots + \lfloor n/25 \rfloor) + \ldots$$

   It follows that

   $$\lim_{n\to\infty} \frac{f(n)}{n^2} = \frac{1/10}{1 - 1/5} = 1/8.$$

10. Let $\{f(n)\}$ be a strictly increasing sequence of positive integers such that $f(2) = 2$ and $f(mn) = f(m)f(n)$ whenever m and n are relatively prime. Show that $f(n) = n$ for all n. (Putnam '63, B2)

    Answer: Let $f(3) = a$. Then

    $$f(5) \geq a + 2, f(15) \geq a(a + 2) \text{ and } f(18) \geq a^2 + 2a + 3.$$

    On the other hand,

    $$f(6) = 2a, f(5) \leq 2a - 1, f(10) \leq 4a - 2, f(9) \leq 4a - 3 \text{ and } f(18) \leq 8a - 6.$$

    Since $a^2 + 2a + 3 > 8a - 6$ for $a \geq 4$, it follows that $f(3) = 3$. We now show by induction that $f(2^k + 1) = 2^k + 1$. The base case $k = 1$ is already verified. If we assume $f(2^k + 1) = 2^k + 1$, we get $f(2^{k+1} + 2) = 2^{k+1} + 2$, so $f(2^{k+1} + 1) \leq 2^{k+1} + 1$. But $f(n) \geq n$ for all n, so we have equality. Now if $f(n) > n$ for some n with $2^k + 1 \leq n < 2^{k+1} + 1$, we have an immediate contradiction. Thus $f(n) = n$ for all n.