Putnam Training Session 7

Tools: Useful Results

1. **Linearity of Expectation:** If $X_1, X_2, \ldots, X_n$ are random variables over the same sample space and $X = X_1 + X_2 + \ldots + X_n$, then
$$E(X) = E(X_1) + \ldots + E(X_n).$$

2. **Law of Total probability:** $Pr(A) = \sum_n Pr(A|B_n)Pr(B_n)$ where $\{B_n\}$ is a collection of mutually exclusive and exhaustive events.

3. **Discrete Convolution:** If X and Y are independent integer-valued random variables, then
$$\begin{aligned} \sum_{m=-\infty}^{\infty} P(X+Y=m)z^m &= \sum_{m=-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} P(X=k)P(Y=m-k) \right) z^m \\ &= \left(\sum_{k=-\infty}^{\infty} P(X=k)z^k \right) \left(\sum_{k=-\infty}^{\infty} P(Y=k)z^k \right) \end{aligned}$$

Problem Set 8: Probability

1. Suppose that a microbe successfully splits into two copies with probability p or fails to split and dies. What is the probability that a single microbe will give rise to an everlasting colony?

2. Let σ be a permutation of $\{1, 2, \ldots, n\}$. We say that j is a fixed point of σ if $\sigma(j) = j$. Find the expected number of fixed points of a random permutation σ of $\{1, 2, \ldots, n\}$.

3. Let $p(n)$ be the probability that two integers (not necessarily distinct) chosen randomly and uniformly from $\{1, 2, \ldots, n\}$ are relatively prime. Find $\lim_{n\to\infty} p(n)$.

4. Suppose we wish to put positive integer labels on each face of a pair of dice so that the probability of getting sums $2, 3, \ldots, 12$ are in the ratio $1:2:3:4:5:6:5:4:3:2:1$ and the probability of getting any other sum is 0. One way, of course, is to put the numbers 1 through 6 on each die. Find another way.

5. Shanille O'Keal shoots free throws on a basketball court. She hits the first and misses the second, and thereafter, the probability that she hits the next shot is equal to the proportion of shots she has hit so far. What is the probability that she hits exactly 50 of her first 100 shots? (Putnam '02, B1)

6. Let k be a positive integer. Suppose that the integers $1, 2, 3, \ldots, 3k+1$ are written down in random order. What is the probability that at no time during this process, the sum of integers that have been written up to that time is a multiple of 3? (Putnam '07, A3)