Putnam Training Session 7

Tools: Useful Results

1. **Linearity of Expectation:** If $X_1, X_2, \ldots, X_n$ are random variables over the same sample space and $X = X_1 + X_2 + \ldots + X_n$, then

$$E(X) = E(X_1) + \ldots + E(X_n).$$

2. **Law of Total probability:** $Pr(A) = \sum_n Pr(A|B_n)Pr(B_n)$ where $\{B_n\}$ is a collection of mutually exclusive and exhaustive events.

3. **Discrete Convolution:** If X and Y are independent integer-valued random variables, then

$$\begin{aligned}\sum_{m=-\infty}^{\infty} P(X+Y=m)z^m &= \sum_{m=-\infty}^{\infty}\left(\sum_{k=-\infty}^{\infty} P(X=k)P(Y=m-k)\right)z^m \\ &= \left(\sum_{k=-\infty}^{\infty} P(X=k)z^k\right)\left(\sum_{k=-\infty}^{\infty} P(Y=k)z^k\right)\end{aligned}$$

Problem Set 8: Probability
Hints and Solutions

1. Suppose that a microbe successfully splits into two copies with probability p or fails to split and dies. What is the probability that a single microbe will give rise to an everlasting colony?

   Answer: $2-1/p$ if $p \geq 1/2$, 0 otherwise. If q is the required probability, then $q = p(1-(1-q)^2)$, since a successful division leaves us with the problem of determining the probability that either of the two microbes leaves an everlasting colony. Some justification should accompany the elimination of the solution $q = 0$ for $p \geq 1/2$.

2. Let σ be a permutation of $\{1, 2, \ldots, n\}$. We say that j is a fixed point of σ if $\sigma(j) = j$. Find the expected number of fixed points of a random permutation σ of $\{1, 2, \ldots, n\}$.

   Answer: For $1 \leq i \leq n$, let X_i denote the event that $\sigma(i) = i$. Then $E(X_i) = 1/n$. Now let the random variable X denote the number of fixed points of a randomly chosen permutation. Since $X = X_1 + \ldots + X_n$, by the linearity of expectation, we have $E(X) = 1$.

3. Let $p(n)$ be the probability that two integers (not necessarily distinct) chosen randomly and uniformly from $\{1, 2, \ldots, n\}$ are relatively prime. Find $\lim_{n\to\infty} p(n)$.

Answer: $6/\pi^2$. The main idea is to consider $p(n,d)$ in general, the probability that two randomly chosen integers has greatest common divisor d. For fixed d and large n, $p(n,d) = p(n,1)/d^2 + O(1/n^2)$. Since $\sum_{d=1}^{\infty} 1/d^2 = \pi^2/6$, it follows that $\lim_{n\to\infty} p(n) = 6/\pi^2$.

4. Suppose we wish to put positive integer labels on each face of a pair of dice so that the probability of getting sums $2, 3, \ldots, 12$ are in the ratio $1:2:3:4:5:6:5:4:3:2:1$ and the probability of getting any other sum is 0. One way, of course, is to put the numbers 1 through 6 on each die. Find another way.

   Answer: $1, 2, 2, 3, 3, 4$ and $1, 3, 4, 5, 6, 8$. The trick is to observe that

$$p(x) = x + 2x^2 + 2x^3 + x^4 = x(1+x)(1+x+x^2) \text{ and}$$

$$q(x) = x + x^3 + x^4 + x^5 + x^6 + x^8 = x(1+x)(1+x+x^2)(1-x+x^2)^2 \text{ satisfy}$$

$$p(1) = q(1) = 6 \text{ and } p(x)q(x) = (1 + x + x^2 + x^3 + x^4 + x^5 + x^6)^2$$

5. Shanille O'Keal shoots free throws on a basketball court. She hits the first and misses the second, and thereafter, the probability that she hits the next shot is equal to the proportion of shots she has hit so far. What is the probability that she hits exactly 50 of her first 100 shots? (Putnam '02, B1)

   Answer: $1/99$. It is easy to show by induction on n that all numbers from 1 to $n-1$ have equal probability to be the number of hits after n throws.

6. Let k be a positive integer. Suppose that the integers $1, 2, 3, \ldots, 3k+1$ are written down in random order. What is the probability that at no time during this process, the sum of integers that have been written up to that time is a multiple of 3? (Putnam '07, A3)

   Answer: $(k!(k+1)!)/((3k+1)(2k)!)$. For the condition to hold, the numbers themselves don't matter; only their residues modulo 3 do. Furthermore, multiples of 3 have no effect. In the subsequence of numbers that are not multiples of 3, the second number should be congruent to the first modulo 3, and thereafter the congruence classes modulo 3 must alternate. Since we have $k+1$ numbers congruent to 1 modulo 3 and only k numbers congruent to 2 modulo 3, this means we must start with a number congruent to 1 modulo 3. (Note that the first number in the sequence cannot be a multiple of 3.) Since the remaining elements of the sequence can be permuted within their congruence classes in $(k!)^3$ ways, and the multiples of 3 can be placed in $\binom{3k}{k}$ ways, the required probability is

$$\frac{(k+1)(k!)^3\binom{3k}{k}}{(3k+1)!} = \frac{k!(k+1)!}{(3k+1)(2k)!}$$