Putnam Training Session 8

Problem Set 9: Sequences

1. Let a and d be positive integers. Show that the arithmetic progression $a, a+d, a+2d, \ldots$ either contains no perfect square or contains infinitely many perfect squares.

2. Let $\{x_n\}$ be a sequence of real numbers satisfying $x_n = (x_{n-1}+x_{n-2})/2$. Show that the sequence converges, and find the limit in terms of x_0 and x_1.

3. Show that every positive integer can be uniquely expressed as the sum of distinct non-consecutive Fibonacci numbers. (For example, $82 = 55 + 21 + 5 + 1$.)

4. Let $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be infinite sequences of positive integers. Show that there exist distinct indices p and q such that $x_p \geq x_q, y_p \geq y_q$ and $z_p \geq z_q$.

5. Let α and β be positive irrational numbers with $1/\alpha + 1/\beta = 1$. Show that every positive integer belongs to either $A = \{\lfloor\alpha\rfloor, \lfloor 2\alpha\rfloor, \lfloor 3\alpha\rfloor, \ldots\}$ or $B = \{\lfloor\beta\rfloor, \lfloor 2\beta\rfloor, \lfloor 3\beta\rfloor, \ldots\}$, but not both.

6. Let $a_1 = a_2 = 1$ and $a_n = (a_{n-1}^2 + 2)/a_{n-2}$ for $n \geq 3$. Show that a_n is an integer for all n.

7. For each integer $n \geq 0$, let $S(n) = n - m^2$, where m is the largest integer with $m^2 \leq n$. Define a sequence $\{a_k\}$ by $a_0 = A; a_{k+1} = a_k + S(a_k)$. For what positive integers A is $\{a_k\}$ eventually constant? (Putnam '91, B1)

8. Let $T_1 = 2$, and for $n \geq 1$, let $T_{n+1} = T_n^2 - T_n + 1$. Show that $(T_m, T_n) = 1$ if $m \neq n$ and $\sum_{k=1}^{\infty}(1/T_k) = 1$. (Putnam '56, B6)