QUIZ5

Name : **Date:**

This quiz will be used to check your attendance and performance. You can use your class notes and handouts. If you need, it is o.k. to discuss with your classmates.

1.

Claims filed under auto insurance policies follow a normal distribution with mean 19,400 and standard deviation 5,000. What is the probability that the average of 25 randomly selected claims exceeds 20,000 ?

(A) 0.01 (B) 0.15 (C) 0.27 (D) 0.33 (E) 0.45

Show your work in details.

Answer: C: 0.27

Solution: The probability to compute is $P(\overline{X} \geq 20,000)$, where $\overline{X}$ is the mean of a random sample of size 25 from a normal distribution with mean 19,400 and with standard deviation 5,000. By the CLT (or since the sum of independent normals is again normal), $\overline{X}$ has normal distribution with the same mean and with standard deviation $5,000/\sqrt{25} = 1,000$. After standardizing, we get

$$P(\overline{X} \geq 20,000) = P\left(\frac{\overline{X} - 19,400}{1000} \geq \frac{20,000 - 19,4000}{1000}\right)$$
$$= P(Z \geq 0.6) = 1 - \Phi(0.6) = 0.275.$$

2.

In an analysis of healthcare data, ages have been rounded to the nearest multiple of 5 years. The difference between the true age and the rounded age is assumed to be uniformly distributed on the interval from -2.5 years to 2.5 years. The healthcare data are based on a random sample of 48 people. What is the approximate probability that the mean of the rounded ages is within 0.25 years of the mean of the true ages?

(A) 0.14 (B) 0.38 (C) 0.57 (D) 0.77 (E) 0.88

Show your work in details.

Solution: We need to compute $P(|\overline{X}| \leq 0.25)$, where $\overline{X}$ is the mean of a random sample of size 48 from a uniform distribution on $[-2.5, 2.5]$. Obviously this distribution has mean 0, and an easy computation (or using the formula $\sigma^2 = (b-a)^2/12$ for the variance of a uniform distribution on an interval $[a, b]$) shows that the standard deviation is 1.44. By the CLT, $\overline{X}$ has approximately normal distribution with mean 0 and standard deviation $1.44/\sqrt{48} = 0.2078$, so

$$P(|\overline{X}| \leq 0.25) = P\left(\left|\frac{\overline{X} - 0}{0.2078}\right| \leq \frac{0.25}{0.2078}\right)$$
$$\approx P(|Z| \leq 1.2) = \Phi(1.2) - \Phi(-1.2) = 0.769.$$

3.

A charity receives 2025 contributions. Contributions are assumed to be independent and identically distributed with mean 3125 and standard deviation 250. Calculate the approximate 90th percentile for the distribution of the total contributions received.

(A) $6,328,000$ (B) $6,338,000$ (C) $6,343,000$ (D) $6,784,000$ (E) $6,977,000$

Show your work in details.

Answer: C: 6,343,000

Solution: If S denotes the total (i.e., sum) of all contributions received, we need to find x such that, approximately, $P(S \le x) = 0.9$. Now S is the sum of 2025 independent r.v.'s, each with mean $\mu = 3125$ and standard deviation 250. Hence, by the CLT, S is approximately normal with mean $1250 \cdot 2025 = 6328125$ and standard deviation $250 \cdot \sqrt{2025} = 11250$, and so

$$P(S \le x) = P\left(\frac{S - 6328125}{11250} \le \frac{x - 6328125}{11250}\right)$$
$$\approx P\left(Z \le \frac{x - 6328125}{11250}\right) = \Phi\left(\frac{x - 6328125}{11250}\right).$$

Setting this equal to 0.9, we get from the normal table $(x - 6328125)/11250 = 1.28$, and so $x = 6328125 + 11250 \cdot 1.28 = 6342525$.

4.

Two instruments are used to measure the height, h, of a tower. The error made by the less accurate instrument is normally distributed with mean 0 and standard deviation $0.0056h$. The error made by the more accurate instrument is normally distributed with mean 0 and standard deviation $0.0044h$. Assuming the two measurements are independent random variables, what is the probability that their average value is within $0.005h$ of the height of the tower?

(A) 0.38 (B) 0.47 (C) 0.68 (D) 0.84 (E) 0.90

Show your work in details.

Answer: D: 0.84

Solution: If X and Y denote the two measurements, then their average is $A = (1/2)X + (1/2)Y$. We need to compute $P(|A| \leq 0.005h)$. Since A is a linear combination of independent normals, it is normal, and the mean and variance of A can be computed as follows:

$$\begin{aligned} E(A) &= E((1/2)X + (1/2)Y) = 0, \\ \text{Var}(A) &= \text{Var}((1/2)X + (1/2)Y) \\ &= \frac{1}{4}\text{Var}(X) + \frac{1}{4}\text{Var}(Y)) = \frac{1}{4}(0.0056h^2 + 0.0044h^2) = 0.00001268h^2. \end{aligned}$$

Thus, after standardizing A by dividing by $\sqrt{0.00001268^2 h^2} = 0.00356h$, the probability to compute becomes

$$\begin{aligned} P(|A| \leq 0.005h) &= P\left(\left|\frac{A}{0.00356h}\right| \leq \frac{0.005h}{0.00356h}\right) \\ &= P(|Z| \leq 1.4) = \Phi(1.4) - \Phi(-1.4) = 0.838. \end{aligned}$$

5.

For Company A there is a 60% chance that no claim is made during the coming year. If one or more claims are made, the total claim amount is normally distributed with mean 10,000 and standard deviation 2,000. For Company B there is a 70% chance that no claim is made during the coming year. If one or more claims are made, the total claim amount is normally distributed with mean 9,000 and standard deviation 2,000. Assume that the total claim amounts of the two companies are independent. What is the probability that, in the coming year, Company B's total claim amount will exceed Company A's total claim amount?

(A) 0.180 (B) 0.185 (C) 0.217 (D) 0.223 (E) 0.240

Show your work in details.

Answer: D: 0.223

Solution: What makes this problem tricky is the fact that the claim amounts, say X and Y, of the two companies only take effect if there are claims, and that for each company there is a positive probability that there are no claims.

We first carry out the calculation under the assumption that both companies do have claims. The probability that the claim amounts of B exceed those of A then is $P(Y > X)$, or, equivalently, $P(X - Y < 0)$. Since X and Y are independent normal r.v.'s with respective means $10,000$ and $9,000$ and variances $2,000^2$, $X - Y$ has normal distribution with mean $10,000 - 9,000 = 1000$, variance $2 \cdot 2,000^2 = 8,000,000$, and standard deviation $\sqrt{8,000,000} = 2,828$. Converting to standard units, we get

$$\begin{aligned} P(X - Y < 0) &= P\left(\frac{X - Y - 1000}{2828} < \frac{0 - 1000}{2828}\right) \\ &= P(Z < -0.3535) = P(Z > 0.3535) = 1 - P(Z \leq 0.3535) \\ &= 1 - \Phi(0.3535) = 0.362. \end{aligned}$$

To account for the fact that one or both of the companies can have no claims, consider the four cases: (1) A and B both have no claims; (2) A has a claim, B has no claim; (3) A has no claim, B has a claim; and (4) A and B both have claims. These cases are mutually exclusive and exhaust all possibilities. Clearly, the claim amount paid to B exceeds that paid to B if and only if we are in case (2) (which occurs with probability $0.6 \cdot 0.3 = 0.18$) or we are in case (4) (which has probability $0.3 \cdot 0.4 = 0.12$) and the r.v.'s X and Y above satisfy $P(Y > X)$ (which occurs with probability 0.362 by the above calculation). Hence the probability in question is $0.18 + 0.12 \cdot 0.362 = 0.223$.