EX1

Suppose that the cost of maintaining a car is given by a random variable, X, with mean 200 and variance 260. If a tax of 20% is introducted on all items associated with the maintenance of the car, what will the variance of the cost of maintaining a car be?

Solution: The new cost is $1.2X$, so its variance is $\mathrm{Var}(1.2X) = 1.2^2\,\mathrm{Var}(X) = 1.44 \cdot 260 = 374$.

EX2

An insurance policy pays a total medical benefit consisting of a part paid to the surgeon, X, and a part paid to the hospital, Y, so that the total benefit is $X + Y$. Suppose that $\mathrm{Var}(X) = 5,000$, $\mathrm{Var}(Y) = 10,000$, and $\mathrm{Var}(X + Y) = 17,000$.

If X is increased by a flat amount of 100, and Y is increased by 10%, what is the variance of the total benefit after these increases?

Solution: We need to compute $\mathrm{Var}(X + 100 + 1.1Y)$. Since adding constants does not change the variance, this is the same as $\mathrm{Var}(X + 1.1Y)$, which expands as follows:

$$\mathrm{Var}(X + 1.1Y) = \mathrm{Var}(X) + \mathrm{Var}(1.1Y) + 2\,\mathrm{Cov}(X, 1.1Y) = \mathrm{Var}(X) + 1.1^2\,\mathrm{Var}(Y) + 2 \cdot 1.1\,\mathrm{Cov}(X, Y)$$

We are given that $\mathrm{Var}(X) = 5,000$, $\mathrm{Var}(Y) = 10,000$, so the only remaining unknown quantity is $\mathrm{Cov}(X, Y)$, which can be computed via the general formula for $\mathrm{Var}(X + Y)$:

$$\mathrm{Cov}(X, Y) = \frac{1}{2}\left(\mathrm{Var}(X + Y) - \mathrm{Var}(X) - \mathrm{Var}(Y)\right) = \frac{1}{2}(17,000 - 5,000 - 10,000) = 1,000.$$

Substituting this into the above formula, we get the answer:

$$\mathrm{Var}(X + 1.1Y) = 5,000 + 1.1^2 \cdot 10,000 + 2 \cdot 1.1 \cdot 1,000 = 19,520$$

EX3

Given that $E(X) = 5$, $E(X^2) = 27.4$, $E(Y) = 7$, $E(Y^2) = 51.4$ and $\mathrm{Var}(X + Y) = 8$, find $\mathrm{Cov}(X + Y, X + 1.2Y)$.

Solution: By definition,

$$\text{Cov}(X+Y, X+1.2Y) = E((X+Y)(X+1.2Y)) - E(X+Y)E(X+1.2Y).$$

Using the properties of expectation and the given data, we get

$$\begin{aligned} E(X+Y)E(X+1.2Y) &= (E(X)+E(Y))(E(X)+1.2E(Y)) = (5+7)(5+1.2\cdot 7) = 160.8, \\ E((X+Y)(X+1.2Y)) &= E(X^2)+2.2E(XY)+1.2E(Y^2) \\ &= 27.4+2.2E(XY)+1.2\cdot 51.4 = 2.2E(XY)+89.08, \\ \text{Cov}(X+Y, X+1.2Y) &= 2.2E(XY)+89.08-160.8 = 2.2E(XY)-71.72 \end{aligned}$$

To complete the calculation, it remains to find $E(XY)$. To this end we make use of the still unused relation $\text{Var}(X+Y) = 8$:

$$\begin{aligned} 8 = \text{Var}(X+Y) &= E((X+Y)^2) - (E(X+Y))^2 = E(X^2)+2E(XY)+E(Y^2)-(E(X)+E(Y))^2 \\ &= 27.4+2E(XY)+51.4-(5+7)^2 = 2E(XY)-65.2, \end{aligned}$$

so $E(XY) = 36.6$. Substituting this above gives $\text{Cov}(X+Y, X+1.2Y) = 2.2\cdot 36.6 - 71.72 = 8.8$.

EX4

. (Cf. Problem 2.5.21 in Hogg/Tanis) Given that X has moment-generating function

$$M(t) = \frac{1}{6}e^{-2t} + \frac{1}{3}e^{-t} + \frac{1}{4}e^{t} + \frac{1}{4}e^{2t},$$

find $P(|X| \le 1)$.

Solution: Comparing the given formula for $M(t)$ with the general formula for a the mgf of a discrete distribution, $M_X(t) = \sum_x f(x)e^{tx}$, where the sum is over all values x of X, we see that X must have values -2, -1, 1, and 2, with probabilities 1/6, 1/3, 1/4, and 1/4, respectively. Thus, $P(|X| \le 1) = P(X=-1) + P(X=1) = 1/3 + 1/4 = 7/12$.

EX5

. (May 2007 Exam) Suppose that $M(t)$ is a moment-generating function of some random variable. Which of the following are moment-generating functions of some (other) random variables?
(i) $M(t)M(5t)$; (ii) $2M(t)$; (iii) $e^{-t}M(t)$.

Solution: The second function, $2M(t)$, can be eliminated since at $t = 0$ it equals $2M(0) = 2 \cdot 1 = 2$, whereas any mgf must be equal to 1 at $t = 0$.

The other two functions, however, are mgfs of suitable random variables. This hinges on the following facts:

(1) The product of any two mgf's is again an mgf (for some r.v.).

(2) If $M(t)$ is an mgf, then so is $M(ct)$ for any non-zero constant c.

(3) The function e^{-t} is an mgf.

To see (1), suppose $M_1(t)$ and $M_2(t)$ are the mgf's of random variable X_1 and X_2. One can, without loss of generality, assume that X_1 and X_2 are independent. Then $X_1 + X_2$ has mgf $M_1(t)M_2(t)$.

To see (2), let $M(t)$ be the mgf of X, and let $Y = cX$. Then $M_Y(t) = E(e^{tY}) = E(e^{tcX}) = M(tc) = M(ct)$, so the function $M(ct)$ is the mgf of Y.

To see (3), simply take X to be the random variable that takes on the single value -1 with probability 1. Then X has mgf $1 \cdot e^{-t}$.