UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN
Actuarial Science Program
DEPARTMENT OF MATHEMATICS

Math 370 (Z) Prof. Rick Gorvett
Exam 2/FM Preparation

More Annuities and Perpetuities
Review Problems

Topic A: Continuous Annuities

$$a_{\overline{n}|}^{(m)} \xrightarrow[m\to\infty]{} \bar{a}_{\overline{n}|i} = \int_0^n v^t\,dt \quad (\$1 \text{ paid per conv. period})$$

$$\bar{a}_{\overline{n}|i} = \frac{1-v^n}{\delta} = \frac{1-e^{-n\delta}}{\delta}$$

$$\bar{s}_{\overline{n}|i} = \frac{(1+i)^n - 1}{\delta} = \frac{e^{n\delta}-1}{\delta} = \int_0^n (1+i)^t\,dt$$

(1) Payments are made to an account at a continuous rate of $(8k + tk)$, where $0 \le t \le 10$. Interest is credited at a force of interest $\delta_t = 1/(8+t)$. After 10 years, the account is worth 20,000. Calculate k.

(A) 111
(B) 116
(C) 121
(D) 126
(E) 131

(2005 sample question # 21)

$$20{,}000 = \int_0^{10} (8k + tk)(1+i)^{10-t}\,dt$$

Note: $(1+i)^{10-t} = e^{\int_t^{10}\left(\frac{1}{8+n}\right)dn} = e^{\ln(8+n)\big|_t^{10}} = e^{\ln(18)-\ln(8+t)} = \frac{18}{8+t}$.

$$\Rightarrow 20{,}000 = \int_0^{10}(8k+tk)\left(\frac{18}{8+t}\right)dt = \int_0^{10} k\cdot 18\,dt$$
$$= 18kt\Big|_0^{10} = 180k \Rightarrow k = \underline{111.11}$$

Topic B: Arithmetically-Varying Annuities

$$PV = Pa_{\overline{n}|} + Q\left(\frac{a_{\overline{n}|} - nv^n}{i}\right)$$

$$Accumulated\ Value = Ps_{\overline{n}|} + Q\left(\frac{s_{\overline{n}|} - n}{i}\right)$$

$$(Ia)_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{i} \qquad (Is)_{\overline{n}|} = \frac{\ddot{s}_{\overline{n}|} - n}{i}$$

$$(Da)_{\overline{n}|} = \frac{n - a_{\overline{n}|}}{i} \qquad (Ds)_{\overline{n}|} = \frac{n(1+i)^n - s_{\overline{n}|}}{i}$$

(2) The present value of a series of 50 payments starting at 100 at the end of the first year and increasing by 1 each year thereafter is equal to X. The annual effective rate of interest is 9%. Calculate X.

$$(1)\quad PV = 100 \cdot a_{\overline{50|}} + 1 \cdot \left(\frac{a_{\overline{50|}} - 50v^{50}}{.09}\right) = \underline{\underline{1210.49}}$$

(A) 1165
(B) 1180
(C) 1195
(D) 1210
(E) 1225

$$(2)\quad PV = 99 \cdot a_{\overline{50|}} + (Ia)_{\overline{50|}} = \underline{\underline{1210.49}}$$

(May 2005 Exam 2/FM, question # 9)

(3) An annuity-immediate pays 20 per year for 10 years, then decreases by 1 per year for 19 years. At an annual effective interest rate of 6%, the present value is equal to X. Calculate X.

$$(1)\quad 20 \cdot a_{\overline{10|}} + v^{10} \cdot (Da)_{\overline{19|}} = \underline{\underline{220.18}}$$

$$(2)\quad 20 \cdot a_{\overline{9|}} + v^{9} \cdot \left[20 \cdot a_{\overline{20|}} - 1 \cdot \left(\frac{a_{\overline{20|}} - 20v^{20}}{.06}\right)\right] = \underline{\underline{220.18}}$$

(A) 200
(B) 205
(C) 210
(D) 215
(E) 220

(May 2005 Exam 2/FM, question # 14)

(4) At an annual effective interest rate of i, the present value of a perpetuity-immediate starting with a payment of 200 in the first year and increasing by 50 each year thereafter is 46,530. Calculate i.

$$PV = \frac{200}{i} + \frac{50}{i^2} = 46{,}530$$

$$\Rightarrow 46{,}530 i^2 - 200 i - 50 = 0$$

(A) 3.25%
(B) 3.50%
(C) 3.75%
(D) 4.00%
(E) 4.25%

(May 2005 Exam 2/FM, question # 17)

$$\Rightarrow i = \frac{200 \pm \sqrt{40{,}000 + 9{,}306{,}000}}{93{,}060} = \underline{\underline{.0350}}$$

(5) Megan purchases a perpetuity-immediate for 3250 with annual payments of 130. At the same price and interest rate, Chris purchases an annuity-immediate with 20 annual payments that begin at amount P and increase by 15 each year thereafter. Calculate P.

MEGAN: $3250 = \frac{130}{i} \Rightarrow i = .04$

CHRIS: $3250 = P \cdot a_{\overline{20|}} + 15\left(\frac{a_{\overline{20|}} - 20v^{20}}{.04}\right)$

(A) 90
(B) 116
(C) 131
(D) 176
(E) 239

(November 2005 Exam 2/FM, question # 12)

$$\Rightarrow P = \underline{\underline{116}}$$

(6) The present value of a 25-year annuity-immediate with a first payment of 2500 and decreasing by 100 each year thereafter is X. Assuming an annual effective interest rate of 10%, calculate X.

$$X = 2500 \cdot a_{\overline{25}|} - 100\left(\frac{a_{\overline{25}|} - 25v^{25}}{.10}\right) = \underline{15{,}922.96}$$

$$\left(\text{or: } X = 100 \cdot (Da)_{\overline{25}|}\right)$$

(A) 11,346
(B) 13,615
(C) 15,923
(D) 17,396
(E) 18,112

(November 2005 Exam 2/FM, question # 23)

(7) Olga buys a 5-year increasing annuity for X. Olga will receive 2 at the end of the first month, 4 at the end of the second month, and for each month thereafter the payment increases by 2. The nominal interest rate is 9% convertible quarterly. Calculate X.

$$j = \left(1 + \frac{.09}{4}\right)^{1/3} - 1 = .007444$$

$$X = 2 \cdot a_{\overline{60}|j} + 2\left(\frac{a_{\overline{60}|j} - 60v_j^{60}}{j}\right) = \underline{2729.21}$$

(A) 2680
(B) 2730
(C) 2780
(D) 2830
(E) 2880

(2005 sample question # 18)

Topic C: Geometrically-Varying Annuities

$$\boxed{PV = \frac{1 - \left(\frac{1+k}{1+i}\right)^n}{i-k}}$$

(8) Matthew makes a series of payments at the beginning of each year for 20 years. The first payment is 100. Each subsequent payment through the tenth year increases by 5% from the previous payment. After the tenth payment, each payment decreases by 5% from the previous payment. Calculate the present value of these payments at the time the first payment is made using an annual effective rate of 7%.

$$PV = 100\left[\frac{1 - \left(\frac{1.05}{1.07}\right)^{10}}{.07 - .05}\right](1.07) + 100(1.05)^9(.95)\left[\frac{1 - \left(\frac{.95}{1.07}\right)^{10}}{.07 + .05}\right]\left(\frac{1}{1.07}\right)^9$$

(A) 1375
(B) 1385
(C) 1395
(D) 1405
(E) 1415

(November 2005 Exam 2/FM, question # 8)

$$= 1384.65$$

NOTE:

0	1	9	10	19	20
100	105	155.13	147.38	92.88	

(9) An insurance company has an obligation to pay the medical costs for a claimant. Average annual claims costs today are \$5,000, and medical inflation is expected to be 7% per year. The claimant is expected to live an additional 20 years. Claim payments are made at yearly intervals, with the first claim payment to be made one year from today. Find the present value of the obligation if the annual interest rate is 5%.

(A) 87,932
(B) 102,514
(C) 114,611
(D) 122,634
(E) Cannot be determined

$$PV = \frac{5000(1.07)}{1.05} + \frac{5000(1.07)^2}{(1.05)^2} + \dots + \frac{5000(1.07)^{20}}{(1.05)^{20}}$$

$$= 5000\left[\left(\frac{1.07}{1.05}\right) + \dots + \left(\frac{1.07}{1.05}\right)^{20}\right]$$

(2005 sample question # 31)

$$\Rightarrow 5000\left(\frac{x - x^{21}}{1 - x}\right) = \underline{\underline{122,633.6}}$$

$$\left(x = \frac{1.07}{1.05}\right)$$

(10) A perpetuity-immediate pays 100 per year. Immediately after the fifth payment, the perpetuity is exchanged for a 25-year annuity-immediate that will pay X at the end of the first year. Each subsequent annual payment will be 8% greater than the preceding payment. The annual effective rate of interest is 8%. Calculate X.

(A) 54
(B) 64
(C) 74
(D) 84
(E) 94

$$@\ t=5:\ \frac{100}{.08} = vX + v^2X(1.08) + v^3X(1.08)^2 + \dots$$

$$\Rightarrow 1250 = \frac{1}{1.08}\left[X + X\left(\frac{1.08}{1.08}\right) + \dots + X\left(\frac{1.08}{1.08}\right)^{24}\right]$$

(2005 sample question # 11)

$$= \frac{25X}{1.08} \Rightarrow X = \underline{\underline{54}}$$

(11) Mike buys a perpetuity-immediate with varying annual payments. During the first 5 years, the payment is constant and equal to 10. Beginning in year 6, the payments start to increase. For year 6 and all future years, the current year's payment is K% larger than the previous year's payment. At an annual effective interest rate of 9.2%, the perpetuity has a present value of 167.50. Calculate K, given K < 9.2.

(A) 4.0
(B) 4.2
(C) 4.4
(D) 4.6
(E) 4.8

$$167.50 = 10 \cdot a_{\overline{5}|.092} + v^5\left[v(10)(1+k) + v^2(10)(1+k)^2 + \dots\right]$$

$$= 38.695501 + \left(\frac{1}{1.092}\right)^5\left[10\left(\frac{1+k}{1.092}\right) + \dots + 10\left(\frac{1+k}{1.092}\right)^n + \dots\right]$$

(2005 sample question # 14)

$$\Rightarrow 167.50 = 38.695501 + 6.44\left[\frac{1+k}{.092-k}\right]$$

$$\Rightarrow k = \underline{\underline{.0400}}$$