1.

Upon arrival at a hospital's emergency room, patients are categorized according to their condition as critical, serious, or stable. In the past year:

(i) 10% of the emergency room patients were critical;

(ii) 30% of the emergency room patients were serious;

(iii) the rest of the emergency room patients were stable;

(iv) 40% of the critical patients died;

(vi) 10% of the serious patients died; and

(vii) 1% of the stable patients died.

Given that a patient survived, what is the probability that the patient was categorized as serious upon arrival?

(A) 0.06 (B) 0.29 (C) 0.30 (D) 0.39 (E) 0.64

Answer: B: 0.29

Hint/Solution: This is a standard Bayes' Rule exercise. Let C, S, T denote the events "critical", "serious", and "stable", and let D denote the event "patient dies". We need to compute $P(S|D')$.

Applying Bayes' Rule with C, S, T as the partition of the sample space, and substituting the given data, we get

$$\begin{aligned} P(S|D') &= \frac{P(D'|S)P(S)}{P(D'|S)P(S) + P(D'|C)P(C) + P(D'|T)P(T)} \\ &= \frac{(1-0.1)0.3}{(1-0.1)0.3 + (1-0.4)0.1 + (1-0.01)0.6} = 0.292. \end{aligned}$$

(Note the use here of the complement formula for conditional probabilities: $P(D'|C) = 1 - P(D|C) = 1 - 0.1$, etc.)

2.

A public health researcher examines the medical records of a group of 937 men who died in 1999 and discovers that 210 of the men died from causes related to heart disease. Moreover, 312 of the 937 men had at least one parent who suffered from heart disease, and, of these 312 men, 102 died from causes related to heart disease. Determine the probability that a man randomly selected from this group died of causes related to heart disease, given that neither of his parents suffered from heart disease.

(A) 0.115 (B) 0.173 (C) 0.224 (D) 0.327 (E) 0.514

Answer: B: 0.173

Hint/Solution: Let A denote the set of those in the group who died from causes related to heart disease, and B the set of those who had a parent with heart disease. We need to compute $P(A|B')$. From the given data we have

$$P(A) = \frac{210}{937} = 0.224, \quad P(B) = \frac{312}{937} = 0.333, \quad P(A \cap B) = \frac{102}{937} = 0.109.$$

Thus,

$$P(A|B') = \frac{P(A \cap B')}{P(B')} = \frac{P(A) - P(A \cap B)}{1 - P(B)} = \frac{0.224 - 0.109}{1 - 0.333} = 0.173.$$

3.

An insurance company examines its pool of auto insurance customers and gathers the following information:

(i) All customers insure at least one car.

(ii) 70% of the customers insure more than one car.

(iii) 20% of the customers insure a sports car.

(iv) Of those customers who insure more than one car, 15% insure a sports car.

Calculate the probability that a randomly selected customer insures exactly one car and that car is not a sports car.

(A) 0.13 (B) 0.21 (C) 0.24 (D) 0.25 (E) 0.30

Answer: B: 0.21

Hint/Solution: The tricky part here is the proper interpretation of the given data and the question asked. Letting A denote the event "insures more than one car" and B the event "insures a sports car", we need to compute $P(A' \cap B')$. (Note that the complement to A, "at most one car", is equivalent to "exactly one car", by the assumption that all customers insure at least one car.)

In terms of this notation, the given data translates into $P(A) = 0.7$, $P(B) = 0.2$, $P(B|A) = 0.15$, and from this we deduce $P(A \cap B) = P(B|A)P(A) = 0.105$. We now have everything at hand to compute the requested probability:

$$\begin{aligned} P(A' \cap B') &= 1 - P(A \cup B) = 1 - P(A) - P(B) + P(A \cap B) \\ &= 1 - 0.7 - 0.2 + 0.105 = 0.205. \end{aligned}$$

4.

An auto insurance company has $10,000$ policyholders. Each policyholder is classified as

(i) young or old;

(ii) male or female; and

(iii) married or single.

Of these policyholders, 3000 are young, 4600 are male, and 7000 are married. The policyholders can also be classified as 1320 young males, 3010 married males, and 1400 young married persons. Finally, 600 of the policyholders are young married males. How many of the company's policyholders are young, female, and single?

(A) 280 (B) 423 (C) 486 (D) 880 (E) 896

Answer: D: 880

Hint/Solution: If we consider the sets "young", "male" and "married", we want to count the part of the "young" set that is outside the other two sets. Drawing a Venn diagram, we see that this count is given by

$$\begin{aligned} &\#\{\text{young}\} \quad - \quad \#\{\text{young and married}\} \quad - \quad \#\{\text{young and male}\} \\ &\qquad + \quad \#\{\text{young and married and male}\} \\ &= 3000 - 1320 - 1400 + 600 = 880. \end{aligned}$$

5.

Among a large group of patients recovering from shoulder injuries, it is found that 22% visit both a physical therapist and a chiropractor, whereas 12% visit neither of these. The probability that a patient visits a chiropractor exceeds by 0.14 the probability that a patient visits a physical therapist. Determine the probability that a randomly chosen member of this group visits a physical therapist.

(A) 0.26 (B) 0.38 (C) 0.40 (D) 0.48 (E) 0.62

Answer: D: 0.48

Hint/Solution: Let T denote the event "visits a physical therapist", and C the event "visits a chiropractor". We need to compute $P(T)$. We are given

$$P(T \cap C) = 0.22, \quad P(T' \cap C') = 0.12, \quad P(C) = P(T) + 0.14.$$

From the first and third equations we deduce

$$\begin{aligned} P(T \cup C) &= P(T) + P(C) - P(T \cap C) \\ &= P(T) + (P(T) + 0.14) - 0.22 = 2P(T) - 0.08. \end{aligned}$$

On the other hand, from the second equation and De Morgan's Law, we get

$$P(T \cup C) = 1 - P(T' \cap C') = 1 - 0.12 = 0.88.$$

Hence $2P(T) - 0.08 = 0.88$, so $P(T) = (0.88 + 0.08)/2 = 0.48$.

6.

An insurance company sells a one-year automobile policy with a deductible of 2. The probability that the insured will incur a loss is 0.05. If there is a loss, the probability of a loss of amount N is K/N, for $N = 1, \ldots, 5$ and K a constant. These are the only possible loss amounts and no more than one loss can occur. Determine the net premium for this policy.

(A) 0.031 (B) 0.066 (C) 0.072 (D) 0.110 (E) 0.150

Answer: A: 0.031

Solution: The "net premium" is the premium (per policy) at which the company breaks even. This is equal to the expected insurance payout (per policy). Taking into account the deductible, the payout is 1 if $N = 3$; 2 if $N = 4$; 3 if $N = 5$; and 0 in all other cases. Thus, the expected payout is

$$1P(N = 3) + 2P(N = 4) + 3P(N = 5) = 1\frac{K}{3} + 2\frac{K}{4} + 3\frac{K}{5} = \frac{43}{30}K.$$

To determine the constant K, we use the fact that the (overall) probability of a loss is 0.05. Thus, the sum of the probabilities for the possible loss amounts has to equal 0.05:

$$0.05 = \sum_{N=1}^{5} \frac{K}{N} = \frac{137}{60}K.$$

Hence $K = 3/137$, and the above expectation betcomes $(43/30)(3/137) = 0.03138$.

7.
A probability distribution of the claim sizes for an auto insurance policy is given in the table below:

Claim size	Probability
20	0.15
30	0.10
40	0.05
50	0.20
60	0.10
70	0.10
80	0.30

What percentage of the claims are within one standard deviation of the mean claim size?

(A) 45% (B) 55% (C) 68% (D) 85% (E) 100%

Answer: A: 0.45

Solution: A routine, but rather lengthy computation: First find $\mu = E(X)$, $E(X^2)$, $\mathrm{Var}(X)$, and σ. Then add up the probabilities for those x-values that satisfy $|x - \mu| \le \sigma$.

8.
An actuary has discovered that policyholders are three times as likely to file two claims as to file four claims. If the number of claims filed has a Poisson distribution, what is the variance of the number of claims filed?

(A) $1/\sqrt{3}$ (B) 1 (C) $\sqrt{2}$ (D) 2 (E) 4

Answer: D: 2

Solution: Let X denote the number of claims. We are given that X has Poisson distribution and that $P(X = 2) = 3P(X = 4)$. Substitute the formula for a Poisson p.m.f., $P(X = x) = e^{-\lambda}\lambda^x/x!$ into this equation to determine λ. Then use the fact that $\sigma^2 = \lambda$ for a Poisson distribution with parameter λ.

9.

A small commuter plane has 30 seats. The probability that any particular passenger will not show up for a flight is 0.10, independent of other passengers. The airline sells 32 tickets for the flight. Calculate the probability that more passengers show up for the flight than there are seats available.

(A) 0.0042 (B) 0.0343 (C) 0.0382 (D) 0.1221 (E) 0.1564

Answer: E: 0.1564

Solution: Consider the 32 tickets as 32 Bernoulli (success/failure) trials with success meaning that the passenger holding the ticket shows up (so that $p = 0.9$. The probability to compute is then that of getting 31 or 32 successes in 32 such trials, which is given by the binomial distribution.

10.

The number of injury claims per month is modeled by a random variable N with

$$P(N = n) = \frac{1}{(n+1)(n+2)}, \quad \text{where } n \geq 0.$$

Determine the probability of at least one claim during a particular month, given that there have been at most four claims during that month.

(A) 1/3 (B) 2/5 (C) 1/2 (D) 3/5 (E) 5/6

Answer: B: 2/5

Solution: This is a very easy problem: We need to compute $P(N \geq 1 | N \leq 4)$, which is the same as $P(1 \leq N \leq 4)/P(N \leq 4)$. Now,

$$P(1 \leq N \leq 4) = \sum_{n=1}^{4} P(N = n) = \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} = \frac{1}{3},$$

$$P(N \leq 4) = \sum_{n=0}^{4} P(N = n) = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} = \frac{5}{6}.$$

Dividing the first probability by the second gives the answer: $(1/3)/(5/6) = 2/5$.

11.

An insurance company issued insurance policies to 32 independent risks. For each policy, the probability of a claim is 1/6. The benefit given that there is a claim has probability density function

$$f(y) = \begin{cases} 2(1-y), & 0 < y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Calculate the expected value of total benefits paid.

(A) $\frac{16}{9}$ (B) $\frac{8}{3}$ (C) $\frac{32}{9}$ (D) $\frac{16}{3}$ (E) $\frac{32}{3}$

Answer: A: 16/9

Hint/Solution: The expected value of a single benefit is $E(Y) = \int_0^1 yf(y)dy = \int_0^1 (2y - 2y^2)dy = 1/3$. The expected number of claims is $32 \cdot (1/6) = 16/3$. Multiplying these two numbers gives the expected value of all benefits paid, $(1/3)(16/3) = 16/9$.

12.

A large company has determined that the function

$$f(x) = 3x^2, \quad 0 \leq x \leq 1,$$

serves as the payroll density function. That is, the distribution of payroll, $F(x)$, which is the proportion of total payroll earned by the lowest paid fraction x of employees, $0 \leq x \leq 1$, relates to $f(x)$ in the same way that probability distributions and probability densities relate. Gini's index, G, defined as

$$G = 2\int_0^1 |x - F(x)|dx$$

is a measure of how evenly payroll is distributed among all employees. Calculate G for this large company.

(A) 0.2 (B) 0.4 (C) 0.5 (D) 0.8 (E) 1.0

Answer: C: 0.5

Hint/Solution: This reduces to an easy integration problem. First compute $F(x)$ by integrating $f(x)$, then compute the integral in the definition of G.

13.

An insurance company's monthly claims are modeled by a continuous, positive random variable X, whose probability density function is proportional to $(1+x)^{-4}$, where $0 < x < \infty$. Determine the company's expected monthly claims.

(A) $\frac{1}{6}$ (B) $\frac{1}{3}$ (C) $\frac{1}{2}$ (D) 1 (E) 3

Answer: C: 1/2

Hint/Solution: We have $f(x) = c(1+x)^{-4}$ for $0 < x < \infty$, where c is a constant. First determine the proportionality constant c by setting $\int_0^\infty c(1+x)^{-4}$ equal to 1 and solving for c. This gives $c = 3$. Then compute the integral $E(X) = \int_0^\infty xf(x)dx = \int_0^\infty 3x(1+x)^{-4}dx$ by substituting $u = 1+x$, $du = dx$.

14.

An insurance policy reimburses a loss up to a benefit limit of 10. The policyholder's loss, Y, follows a distribution with density function

$$f(y) = \begin{cases} 2y^{-3} & \text{for } y > 1, \\ 0 & \text{otherwise.} \end{cases}$$

What is the expected value of the benefit paid under the insurance policy?

(A) 1.0 (B) 1.3 (C) 1.8 (D) 1.9 (E) 2.0

Answer: D: 1.9

Hint/Solution: The benefit, X, is given by $X = Y$ if $Y \leq 10$, and by $X = 10$ if $Y > 10$. Thus,

$$E(X) = \int_1^{10} y \cdot 2y^{-3}dy + \int_{10}^{\infty} 10 \cdot 2y^{-3}dy = 2\left(1 - \frac{1}{10}\right) + 10 \cdot 10^{-2} = 1.9$$

15.

An insurance company sells an auto insurance policy that covers losses incurred by a policyholder, subject to a deductible of 100. Losses incurred follow an exponential distribution with mean 300. What is the 95th percentile of actual losses that exceed the deductible?

(A) 600 (B) 700 (C) 800 (D) 900 (E) 1000

Hint/Solution: The main difficulty here is the correct interpretation of the "95th percentile of actual losses that exceed the deductible". The proper interpretation involves a conditional probability: we seek the value x such that the conditional probability that the loss is at most x, given that it exceeds the deductible, is 0.95, i.e., that $P(X \leq x | X \geq 100) = 0.95$, where X denotes the loss. By the complement formula for conditional probabilities, this is equivalent to $P(X \geq x | X \geq 100) = 0.05$. Since X is exponentially distributed with mean 300, we have $P(X \geq x) = e^{-x/300}$, so for $x > 100$,

$$P(X \geq x | X \geq 100) = \frac{P(X \geq x)}{P(X \geq 100} = \frac{e^{-x/300}}{e^{-100/300}} = e^{-(x-100)/300}.$$

Setting this equal to 0.05 and solving for x, we get $(x - 100)/300 = -\ln(0.05)$, so $x = -300 \ln(0.05) + 100 = 1000$.