Math 241, Spring 2007, Merit Worksheet 11

1. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ assuming that $z = f(x, y)$ satisfies $x^3 + y^3 + z^3 = xyz$.

2. Let $f(x, y, z) = x \cos y + z^2$. Let P be the point $(3, \pi/3, 5)$.

   (a) Find $\nabla f(P)$.

   (b) Find the directional derivative of f in the direction $\vec{j}$.

   (c) Find the directional derivative of f in the direction $\vec{i} + \vec{j} + \vec{k}$.

3. Look at the contour plot on the next page. At which point will the gradient vector have the largest magnitude? Be most parallel to $\vec{j}$?

   $(0, 2), (-4, -4), (0, 0), (6, -2)$.

4. What is the maximal value that a directional derivative of $f(x, y) = \frac{2x}{x-y}$ can assume at the point $(3, 1)$?

5. The temperature at a point (x, y) on a metal plate is given by $T(x, y) = 100 - x^2 - 2y^2$. A heat-seeking particle starts at the point $(4, 2)$ and moves at each instant in the direction of maximum temperature increase.

   (a) In what direction does the particle move initially?

   (b) Draw some level curves of T in the xy-plane and sketch the path followed by the particle.

   (c) What is the relationship between the tangent vectors to the curve and the gradient vectors of T?

6. Let f be a function of two variables that has continuous partial derivatives and consider the points $A(1, 3), B(3, 3), C(1, 7)$ and $D(6, 15)$. The directional derivative of f at A in the direction towards the point B is 3 and the directional derivative of f at A in the direction towards the point C is 9. What is the directional derivative of f at A in the direction of D?

7. Suppose that you are standing at the point with coordinates $(-100, -100, 430)$ on a hill that has the shape of the graph of $z = 500 - (0.003)x^2 - (0.004)y^2$ (in units of metres). In what (horizontal) direction should you move in order to maintain a constant altitude? To go downhill as quickly as possible?

8. The surfaces $x^2y^2 + 2x + z^3 = 16$ and $3x^2 + y^2 - 2z = 9$ intersect in a curve that passes through the point $P(2, 1, 2)$. Find a tangent vector to the curve of intersection at P.

9. Find an equation for the plane tangent to the paraboloid $z = 2x^2 + 3y^2$ and, simultaneously, parallel to the plane $4x - 3y - z = 10$.

10. The cone with equation $x^2 + y^2 = z^2$ and the plane with equation $2x + 3y + 4z + 2 = 0$ intersect in an ellipse (Figure 12.8.11). Write an equation of the plane normal to this ellipse at the point $P(3, 4, -5)$.

11. Newspaper Hill.

12. Let $f(x, y) = 4x + 2y - x^2 + xy - y^2$.

    (a) Find all points (a, b) where $\nabla f(a, b) =< 0, 0 >$. In which direction(s) is the maximum rate of change of f at these points?

    (b) Find all points on the surface $z = f(x, y)$ where the tangent plane is horizontal.

    (c) Find all the critical points of $f(x, y)$.

    (d) in light of your answers above, what kind of information do critical points give about a function $f(x, y)$?

    (e) Can you classify the critical points? Can you describe the surface $z = f(x, y)$?

Warm-Up Problems for Next Time

1. Quiz Thursday. Exam March 9th. Practice Exam: Wednesday March 7th at 7pm?

2. Find and classify the critical points of the function $f(x, y) = x^3 + y^3 + 3x$.