1. $\nabla f(x,y,z) = \left\langle \frac{1}{2\sqrt{x}}\sqrt{y^2z^3},\ \sqrt{xz^3},\ \frac{3}{2}\sqrt{z}\sqrt{xy^2} \right\rangle$

$\nabla f(2,2,2) = \left\langle \frac{1}{2\sqrt{2}}\sqrt{32},\ \sqrt{16},\ \frac{3}{2}\sqrt{2}\sqrt{8} \right\rangle = \langle 2,4,6 \rangle$

a) Max. dir. deriv $= |\langle 2,4,6\rangle| = \sqrt{2^2+4^2+6^2} = \sqrt{56}$

In direction of $\nabla f = \langle 2,4,6\rangle = 2\vec{i}+4\vec{j}+6\vec{k}$

b) Unit vector in dir. of $\vec{v} = \frac{3}{13}\vec{i} + \frac{12}{13}\vec{j} + \frac{4}{13}\vec{k}$

$\langle 2,4,6\rangle \cdot \left\langle \frac{3}{13}, \frac{12}{13}, \frac{4}{13}\right\rangle = \frac{6+48+24}{13} = \frac{78}{13} = 6.$

2. $f(x,y) = 4xy - 2x^4 - y^2$

$f_x(x,y) = 4y - 8x^3$

$f_y(x,y) = 4x - 2y$

at critical point, $f_x(x,y) = f_y(x,y) = 0.$

$4y - 8x^3 = 0$

$4x - 2y = 0 \Rightarrow y = 2x$

$4(2x) - 8x^3 = 0 \Rightarrow 8x(1-x^2) = 0 \Rightarrow x \in \{0,1,-1\}.$

$x=0 \Rightarrow y=0, \quad x=1 \Rightarrow y=2, \quad x=-1 \Rightarrow y=-2$

$(0,0), (1,2), (-1,-2)$ are the critical points.

Classify: $\Delta = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$

$f_{xx}(x,y) = -24x^2 \qquad f_{yy}(x,y) = -2, \qquad f_{xy} = f_{yx} = 4$

$\Delta = \begin{vmatrix} -24x^2 & 4 \\ 4 & -2 \end{vmatrix} = 48x^2 - 16$

at $(0,0)$, $\Delta = -16 < 0 \Rightarrow$ saddle point at $(0,0)$

at $(1,2), (-1,2)$, $\Delta = 48 - 16 = 32 > 0$, $f_{xx} = -24 < 0.$

$\Rightarrow$ local maximums at $(1,2)$, $(-1,-2)$.

3. Using Lagrange multipliers:
with $f(x,y,z)=x^2+y^2+z^2$ (distance to $(0,0,0)$)^2
and constraint $xyz-8=0.$ $(g(x,y,z))$

$\nabla f = \lambda \nabla g$

$$\left.\begin{array}{l}\lambda yz = 2x\\ \lambda xz = 2y\\ \lambda yx = 2z\\ xyz=8\end{array}\right\} \Rightarrow \left.\begin{array}{l}\lambda xyz = 2x^2\\ \lambda xyz = 2y^2\\ \lambda xyz = 2z^2\\ xyz=8\end{array}\right\} \Rightarrow \begin{array}{l}8\lambda = 2x^2\\ 8\lambda = 2y^2\\ 8\lambda = 2z^2\\ xyz=8\end{array}$$

Thus $2x^2=2y^2=2z^2.$
But we are interested in $x\geq 0, y\geq 0, z\geq 0$ and $xyz=8$
So $x=y=z$, $x^3=8 \Rightarrow x=2=y=z.$
$(2,2,2)$ is closest point. (Clearly min., not max.)

4. $F(x,y,z)= xy^2+2xyz-e^{xz}-8=0$
$\nabla F = \langle y^2+2yz-ze^{xz},\ 2xy+2xz,\ 2xy-xe^{xz}\rangle$
$\nabla F(1,3,0) = \langle 9+0-0,\ 6+0,\ 6-1e^0\rangle = \langle 9,6,5\rangle.$
Tangent Plane: $9(x-1)+6(y-3)+5(z-0)=0$
at $(1,3,0)$ $9x+6y+5z=27.$

5. $f(x,y,z)=\sqrt{x^2+y^2+z^2}$
Note $f(3,4,12)=13.$

$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$

$$\left.\begin{array}{l}\frac{\partial f}{\partial x} = \frac{x}{\sqrt{x^2+y^2+z^2}}\\ \frac{\partial f}{\partial y} = \frac{y}{\sqrt{x^2+y^2+z^2}}\\ \frac{\partial f}{\partial z} = \frac{z}{\sqrt{x^2+y^2+z^2}}\end{array}\right\} \text{at } (3,4,12) \quad \begin{array}{l}\frac{\partial f}{\partial x} = \frac{3}{13}\\ \frac{\partial f}{\partial y} = \frac{4}{13}\\ \frac{\partial f}{\partial z} = \frac{12}{13}\end{array}$$

$f(3.1,4.2,11.7)-f(3,4,12) \approx \frac{3}{13}(0.1)+\frac{4}{13}(0.2)+\frac{12}{13}(-0.3)$
$f(3.1,4.2,11.7) \approx 13-\frac{2.8}{13} = 12\frac{51}{65}.$

6. Surface clearly opens downward and has a highest point.

$f(x,y) = 4xy - x^4 - y^4$

$f_x(x,y) = 4y - 4x^3$

$f_y(x,y) = 4x - 4y^3$

At critical point, $f_x(x,y) = 0 = f_y(x,y)$

$$\left.\begin{aligned} 4y - 4x^3 &= 0 \\ 4x - 4y^3 &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} & y = x^3 \\ & 4x - 4(x^3)^3 = 0 \end{aligned}$$

$\Rightarrow x - x^9 = 0 \quad \Rightarrow \quad x(1-x^8) = 0 \quad \Rightarrow \quad x(1-x^4)(1+x^4) = 0$

$\Rightarrow x(1-x^2)(1+x^2)(1+x^4) = 0 \quad \Rightarrow \quad x(1-x)(1+x)(1+x^2)(1+x^4) = 0$

$\Rightarrow x \in \{0, -1, 1\}$.

$x = 0 \Rightarrow y = 0$, $\quad x = 1 \Rightarrow y = 1$, $\quad x = -1 \Rightarrow y = -1$

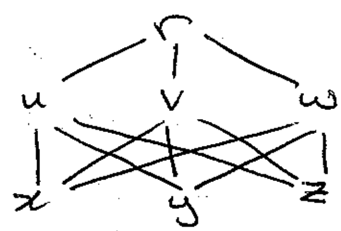
$(0,0) \qquad (1,1) \qquad (-1,-1)$.

Check these 3 points

$f(0,0) = 0, \quad f(1,1) = 4 - 1 - 1 = 2, \quad f(-1,-1) = 4 - 1 - 1 = 2.$

Two highest points $(1,1,2)$, $(-1,-1,2)$ on surface.

7. $r = uvw - u^2 - v^2 - w^2$



$u = y + z$, $v = x + z$, $w = x + y$

$$\frac{\partial r}{\partial x} = \frac{\partial r}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial r}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial r}{\partial w}\frac{\partial w}{\partial x}$$

$= (vw - 2u)\cdot 0 + (uw - 2v)\cdot 1 + (uv - 2w)\cdot 1$

$= uw + uv - 2v - 2w = (u-2)(v+w)$

$$\frac{\partial r}{\partial x} = (y + z - 2)(2x + y + z).$$

8. $xyz = \sin(x+y+z)$. $\qquad$ Differentiate both sides $\frac{\partial}{\partial x}$

$xy\frac{\partial z}{\partial x} + yz = \cos(x+y+z)\left(\frac{\partial z}{\partial x} + 1\right)$ $\qquad$ (LHS) Product Rule & (RHS) Chain Rule

$yz - \cos(x+y+z) = (\cos(x+y+z) - xy)\frac{\partial z}{\partial x}$

$$\frac{yz - \cos(x+y+z)}{\cos(x+y+z) - xy} = \frac{\partial z}{\partial x}$$

9. Perpendicular tangent planes have perpendicular normal vectors.
normal vectors given by gradient vectors

$F(x,y,z) = x^2+y^2+z^2-r^2 = 0$ $\qquad$ $G(x,y,z) = z^2-a^2x^2-b^2y^2=0$

$\nabla F = \langle 2x, 2y, 2z \rangle$ $\qquad$ $\nabla G = \langle -2a^2x, -2b^2y, 2z \rangle$.

Suppose (x_0,y_0,z_0) is a point of intersection of these 2 surfaces.
(So $x_0^2+y_0^2+z_0^2=r^2$ <u>and</u> $z_0^2-a^2x_0^2-b^2y_0^2=0$)

$\nabla F(x_0,y_0,z_0) \cdot \nabla G(x_0,y_0,z_0)$

$= \langle 2x_0, 2y_0, 2z_0 \rangle \cdot \langle -2a^2x_0, -2b^2y_0, 2z_0 \rangle$

$= -4a^2x_0^2 - 4b^2y_0^2 + 4z_0^2$

$= +4(z_0^2 - a^2x_0^2 - b^2y_0^2) = 4(0) = 0.$

So $\nabla F(x_0,y_0,z_0) \perp \nabla G(x_0,y_0,z_0)$

So the tangent planes are perpendicular at every point of intersection
$\Rightarrow$ the surfaces are orthogonal

10. Using Lagrange multipliers: $\nabla f = \lambda \nabla g$

$\langle 3,2,1 \rangle = \lambda \langle 2x, 2y, 2z \rangle$

Note $1 = 2\lambda z \Rightarrow \lambda \neq 0$.

$3 = 2\lambda x \Rightarrow x = \frac{3}{2\lambda}$

$2 = 2\lambda y \Rightarrow y = \frac{2}{2\lambda} = \frac{1}{\lambda}$

$1 = 2\lambda z \Rightarrow z = \frac{1}{2\lambda}$

Also: $x^2+y^2+z^2=1$

$\Rightarrow \frac{9}{4\lambda^2} + \frac{1}{\lambda^2} + \frac{1}{4\lambda^2} = 1$

$\Rightarrow 9 + 4 + 1 = 4\lambda^2$

$14 = 4\lambda^2 \Rightarrow \lambda = \pm\sqrt{\frac{7}{2}}$

Points: $\lambda = \sqrt{\frac{7}{2}} \rightarrow \left(\frac{3}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{1}{\sqrt{14}}\right)$, $f\left(\frac{3}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{1}{\sqrt{14}}\right) = \sqrt{14}$
(Maximum)

$\lambda = -\sqrt{\frac{7}{2}} \rightarrow \left(-\frac{3}{\sqrt{14}}, -\frac{2}{\sqrt{14}}, -\frac{1}{\sqrt{14}}\right)$, $f\left(-\frac{3}{\sqrt{14}}, -\frac{2}{\sqrt{14}}, -\frac{1}{\sqrt{14}}\right) = -\sqrt{14}$
(Minimum)