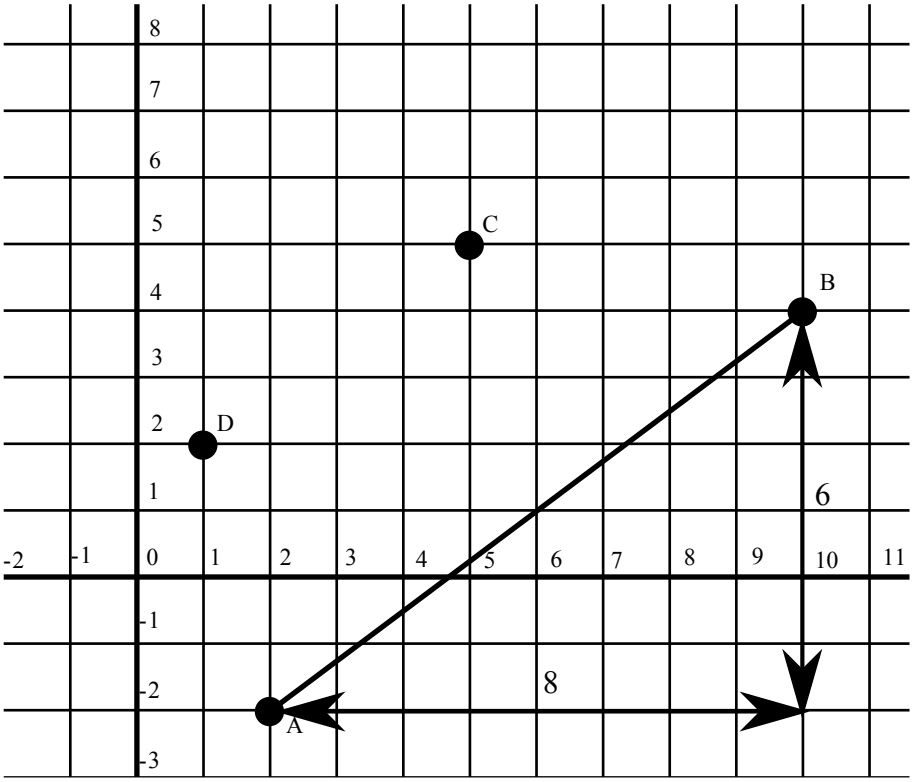
Math 119: Ideas in Geometry, February 1, 2010

Euclidean Geometry

You are standing in the middle of a cornfield in Illinois. A flat plane stretches out in every direction for as far as you can see, with only a little stubble visible through the light layer of snow covering the ground.

1. You walk from A over towards your friend who is standing at B. Draw the path you should take. If you start walking towards B and your friend starts walking towards A, will you meet along the way? If so, where?



There is a unique shortest path between these two points; this is the line segment between these two points. As both you and your friend are walking along this line segment, you will have to have meet each other, presumably at the point $(6, 1)$.

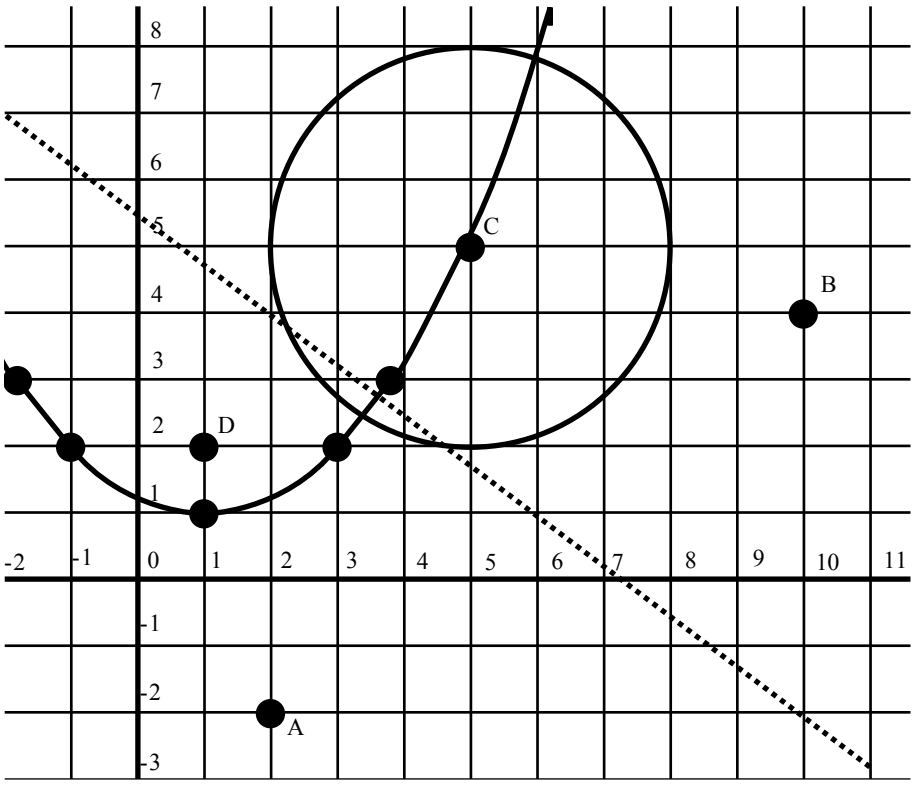
2. What is the distance between the points A and B?

Using either the distance formula (see below) or the Pythagorean Theorem (see the picture above), we see that

$$d_E(A, B) = \sqrt{8^2 + 6^2}.$$

3. What is the formula for the distance between any two points $A = (a_x, a_y)$ and $B = (b_x, b_y)$?

$$d_E(A, B) = \sqrt{(a_x - b_x)^2 + (a_y - b_y)^2}$$



4. Draw the set of points that are distance 3 away from C. (A rough sketch will suffice.)

   This is the definition of the circle centered at C of radius 3.

5. Draw the set of points that are the same distance from D as they are from the x-axis. (A rough sketch will suffice.)

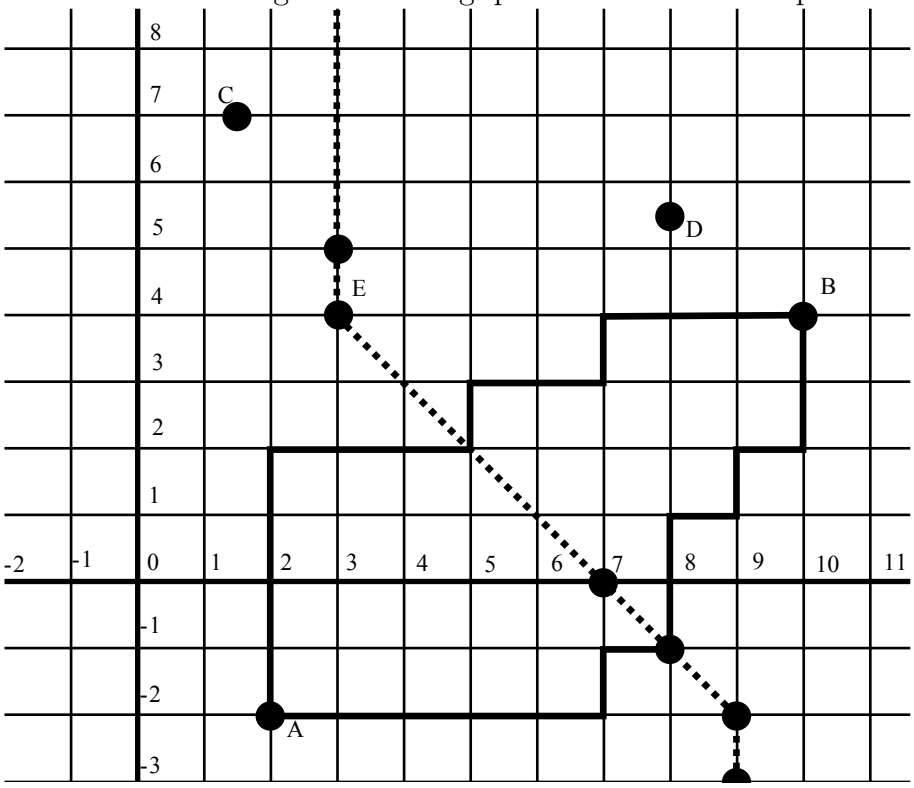
   We have seen this already this semester; this is one way of defining a parabola. For example, the point $(1, 1)$ is 1 away from the point D and 1 away from the x-axis. The points $(-1, 2)$ and $(3, 2)$ are 2 away from the point away from D and 2 away from the y-axis, and so on. See page 12 of the notes for more details.

6. Draw the set of points that are the same distance from both A and B.
   How could one find this set?

   The dotted line is the perpendicular bisector of the line segment from the first question. It is at right angles to the line segment from A to B and passes through its midpoint $(6, 1)$. If you did not know in advance that this was the answer, you could figure it out by following the approach described on page 30 of the notes.

City Geometry and Taxicab Distance

You are in Champaign, walking from point A to your friend's house at point B. Keep this in mind while answering the following questions. Here's a map showing blocks.



1. Your friend is walking to meet you, starting at B and heading towards A. If you start walking (along a shortest path) from A to B, will you meet along the way or could you miss each other? Explain.

   In City Geometry, there can be many different shortest paths between two points. Two of these are shown. If you take the southern route and your friend takes the northern route, you would miss each other.

2. How far do you have to walk to get from point A to point B? Why is this not the same answer as you got last time? What is the distance between C and D?

   No matter what route you take, you have to go 6 blocks north and 8 blocks east to get from A to B. This is a total of 14 blocks (and this is the *taxicab distance* between these two points). This is longer than the Euclidean distance because you cannot walk or drive through the buildings in the middle of the block. This is

   $$|10 - 2| + |4 - (-2)| = 8 + 6 = 14.$$

To get from C to D we need to go $6\frac{1}{2}$ blocks east and $1\frac{1}{2}$ blocks south. This is a total of 8 blocks.

$$\left|1\frac{1}{2} - 8\right| + \left|7 - 5\frac{1}{2}\right| = 6\frac{1}{2} + 1\frac{1}{2} = 8.$$

3. Write down a formula for or explain how to find the distance between any two points $A = (a_x, a_y)$ and $B = (b_x, b_y)$.

   As in the last example, we count the numbers of blocks difference in the north-south direction and the number of blocks difference in the east-west direction and we then add them together. This is

   $$d_T(A, B) = |a_x - b_x| + |a_y - b_y|.$$

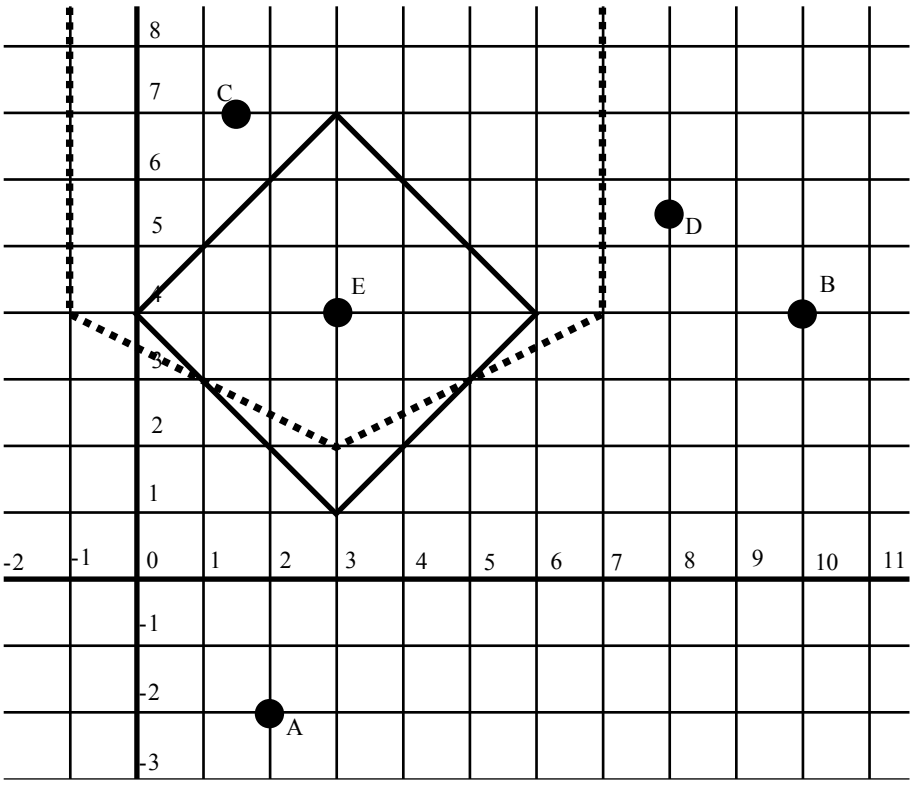
4. You and your friend decide instead to meet at a coffee shop. Feeling like being very fair, you insist that you both walk the same distance. Where can you meet?

   There are many possibilities, e.g.,

   - 7 blocks east of A, which is 6 blocks south and 1 west of B
   - 6 blocks east and 1 north of A, which is 5 blocks south and 2 west of B
   - 5 blocks east and 2 north of A, which is 4 blocks south and 3 west of B
   - All of the above are 7 blocks away from both A and B, but there are also points 8 blocks away from both, 9 blocks away from both, $9\frac{1}{2}$ blocks away from both, etc.. The two points 8 blocks away from both are marked above.
   - Putting all the possibilities together we get the dotted line.

5. Eddie goes for a walk, starting at E. Eddie says he's walked three blocks and will wait for you there but his phone battery dies before he gives you any other information. Where should you look for him? (A rough sketch will suffice.)

   This is the City Geometry version of a circle of radius 3 centered at E. This includes 3 blocks east, or 2 east and 1 south, or 1 east and 2 south, and many others.



6. Draw the set of points that are the same distance from E as they are from the x-axis.

   Compare this with the definition of the parabola on page 12. The dotted lines show a City Geometry version of a parabola. For example, $(5, 3)$ is 3 blocks away from the x-axis and 3 blocks away from E (2 east and 1 south). The point $(-1, 7)$ is 7 blocks away from the x-axis and 7 blocks away from E (4 west and 3 north). See page 32 of the notes for more details.

7. Why might these be called *city geometry* and the *taxicab distance*?

   In a city where the streets are laid out in a rectangular grid, you can only travel east-west and north-south (like a taxicab which only drives on the streets and not through the buildings). In this situation, distances are best described by the numbers of block you travel. This leads to the formula for distance given above.