Math 181: Exam 3

Fall 2008, Tom Cooney

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You have fifty minutes to complete this test. Answer *all* questions.
Calculators and electronic devices are not allowed.

1. (10 points) Give an example of a weighted voting system with 4 voters satisfying *all three* of the following conditions:

   - there is no dictator;
   - there is a dummy voter;
   - there is a voter with veto power.

   In your solution:

   - identify a voter with veto power and explain why this voter has veto power.
   - identify a dummy voter and explain why this voter is a dummy.

   Many different answers are possible. One is:

   $$[q : w_A, w_B, w_C] = [10 : 8, 2, 2, 1]$$

   A has veto power but is not a dictator. All the winning coalitions contain A and thus A has veto power. Or, all the other voters combined have 5 votes, which is less than the quota of 10, and thus A has veto power.

   B, C are ordinary voters.

   D is a dummy voter – the only way for a measure to pass is if A and one or both of B and C support it. D's one vote will not matter in this case. Phrased another way, the winning coalitions containing D are: $ABD, ACD, ABCD$. If we remove D from these, we are still left with winning coalitions: AB, AC, ABC.

2. (10 points) A weighted voting system with four voters has the following winning coalitions (and no others):

$$AB, AC, ABC, ABD, ACD, BCD, ABCD$$

If the quota is 10, determine weights for the four voters. (There is more than one possible correct answer – you only have to find one.)

Clearly, A has the most power. B and C have less power than A but more than D. Using these ideas, we can find suitable weights. Let's try giving A 7 votes – less than the quota but still a significant number of votes. (If this didn't work, we would start again, giving A 6 or 8 votes, say.)

A and B together should have 10 or more votes (as AB is a winning coalition). Let's try giving B 4 votes. Similarly, let's try giving C 4 votes.

We also want BCD to be a winning coalition, so D should have at least 2 votes (4+4+2=10). We do *not* want AD to be a winning coalition, so D should *not* have 3 or more votes (7+3=10, which would make AD into a winning coalition). So try

$$[q : w_A, w_B, w_C, w_D] = [10 : 7, 4, 4, 2].$$

We check the winning coalitions – and they are exactly as desired.

3. (13 points) Consider the weighted voting system

$$[q : w_A, w_B, w_C, w_D, w_E, w_F] = [9 : 3, 2, 2, 2, 2, 2].$$

Calculate the Shapley-Shubik power index for this weighted voting system. Make sure to clearly show/explain how you obtained your answer.

Let's start by describing the permutations for which A is pivotal. A is pivotal in permutations in which A comes after 6, 7, or 8 votes. (It is impossible to get 7 votes here from the other voters, who all have weight 2.)

A is pivotal in the permutations (where X stands for one of the weight 2 voters):

$X_1 X_2 X_3 A X_4 X_5$

$X_1 X_2 X_3 X_4 A X_5$.

There are 5! ways to arrange $X_1, \ldots, X_5$ and so 5! permutations of each type. Thus A is pivotal $2 \times 5 \times 4 \times 3 \times 2 \times 1$ times.

A has Shapley-Shubik power index

$$\frac{\text{Number of permutations in which } A \text{ is pivotal}}{\text{Total number of permutations}} = \frac{2 \times 5 \times 4 \times 3 \times 2 \times 1}{6 \times 5 \times 4 \times 3 \times 2 \times 1} = \frac{1}{3}.$$

The other five voters share the remaining $1 - \frac{1}{3} = \frac{2}{3}$ of the power. So they each have $\frac{2}{3} \div 5 = \frac{2}{15}$ of the power.

The weighted voting system's Shapley-Shubik power index is

$$\left(\frac{1}{3}, \frac{2}{15}, \frac{2}{15}, \frac{2}{15}, \frac{2}{15}, \frac{2}{15}\right).$$

4. (12 points) Consider the weighted voting system

$$[q : w_A, w_B, w_C, w_D] = [8 : 5, 4, 3, 1]$$

Calculate the Banzhaf power index for this weighted voting system. Make sure to clearly show/explain how you obtained your answer.

Winning coalition	Total votes	Extra votes	Critical voters
AB	9	1	A, B
AC	8	0	A, C
ABC	12	4	A
ABD	10	2	A, B
ACD	9	1	A, C
BCD	8	0	B, C, D
ABCD	13	5	–

Count how many times each voter is critical in winning coalitions. Double this to account for blocking coalitions.

The Banzhaf power index of this weighted voting system is

$$(10, 6, 6, 2) .$$

5. (15 points) Consider the weighted voting system

$$[q : w_A, w_B, w_C, w_D, w_E, w_F, w_G, w_H] = [10 : 3, 3, 2, 2, 2, 2, 2, 2].$$

(a) Describe the coalitions in which C, a voter of weight 2, is critical.

Let X stand for one of D, E, F, G, H, a voter of weight 2 different from C.
C is critical in winning coalitions containing C with 0 or 1 extra votes, 10 or 11 total votes.
10 votes:
$ABCX_1$ – there are $C_1^5 = 5$ winning coalitions of this form.
$CX_1X_2X_3X_4$ – there are $C_4^5 = 5$ winning coalitions of this form.
11 votes:
$ACX_1X_2X_3$ – there are $C_3^5 = 10$ winning coalitions of this form.
$BCX_1X_2X_3$ – there are C_3^5=10 winning coalitions of this form.

(b) Calculate the Banzhaf power index of the voter C. (Note: You are *not* asked to calculate the Banzhaf power index of the other voters.)

C is critical in a total of 30 winning coalitions. Double this to account for blocking coalitions. C has Banzhaf power index of 60.

6. (points) Bart and Lisa Simpson are using the adjusted-winner-procedure to divide up their belongings.

Object	Bart	Lisa
Santa's Little Helper	50	30
Snowball II	5	30
Skateboard	30	10
Saxophone	5	20
Be Sharps's album	10	10

Use the adjusted winner procedure to determine a fair allocation of these objects.

Step 1: Give each item to the person who values it most highly.

Bart: Santa's LH, Skateboard for $50 + 30 = 80$ points.

Lisa: Snowball, Saxophone for $30 + 20 = 50$ points.

Step 2: If two people value an item equally, give it to the person who has less points.

Give the album to Lisa. She now has $30 + 20 + 10 = 60$ points.

Step 3: Bart still has more points than Lisa and so should give (part of) an object to her. We look at the point ratios of Bart's objects to decide which object he should give.

Santa's LH: $\frac{50}{30} = \frac{5}{3} = 1.66\ldots$, Skateboard: $\frac{30}{10} = 3$.

So Bart should give (part ownership of) Santa's Little Helper to Lisa. If he gave all of Santa's LH to her, then Bart would have too few points and Lisa too many. So Bart keeps x ownership of Santa's LH and Lisa gets the remaining $1 - x$. We want

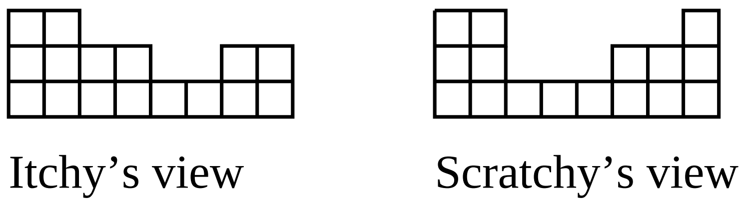
$$\begin{aligned}
\text{Bart's points} &= \text{Lisa's points} \\
30 + 50x &= 30 + 20 + 10 + 30(1 - x) \\
30 + 50x &= 90 - 30x \\
80x &= 60 \\
x &= \frac{60}{80} = \frac{3}{4} \\
1 - x &= 1 - \frac{3}{4} = \frac{1}{4}
\end{aligned}$$

Final allocation:

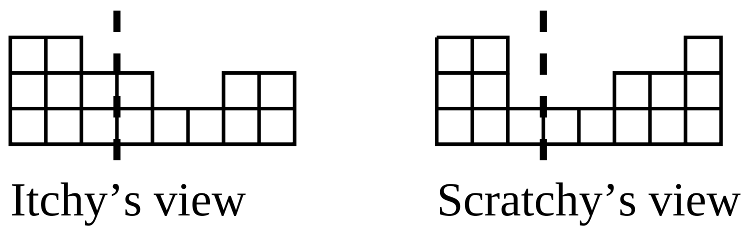
Bart: Skateboard, $\frac{3}{4}$-ownership of Santa's Little Helper.

Lisa: Snowball II, Saxophone, Be Sharps's album, $\frac{1}{4}$-ownership of Santa's Little Helper.

7. (10 points) Itchy and Scratchy are dividing a cake using *divide-and-choose*. Their views of the cake are as follows:

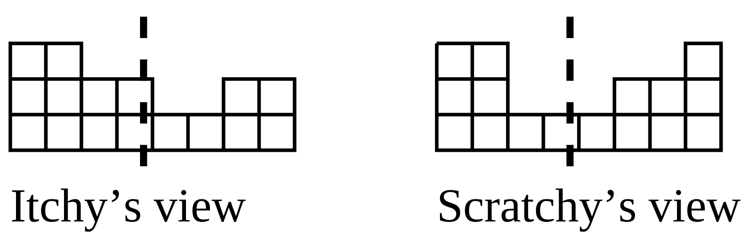


(a) Suppose Itchy is dividing the cake and does *not* know what Scratchy's opinion of the cake is. Where should Itchy cut the cake and how will the cake be divided between them?



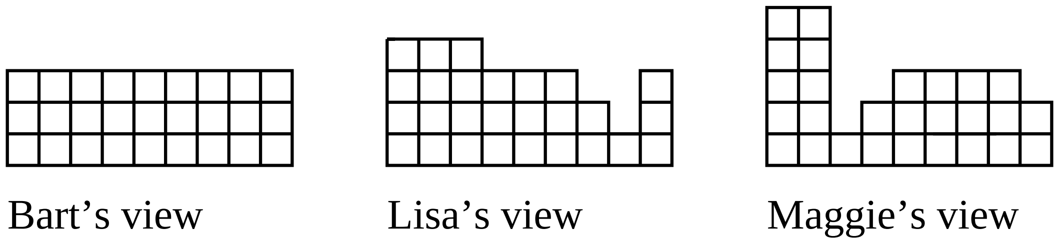
Itchy should divide the cake into two equal pieces to guarantee himself at least that much. Scratchy will choose the piece on the right. Itchy will be left with the piece on the left.

(b) Suppose Itchy is dividing the cake and does know what Scratchy's opinion of the cake is. Where should Itchy cut the cake and how will the cake be divided between them?



Itchy should divide the cake so that Scratchy believes Scratchy is receiving barely more than half the cake. Itchy should cut slightly less than 4 marks across from the left side. Then Scratchy will view the piece on the right as being a bit more than 8 units and will choose the piece on the right. Itchy will be left with the piece on the left, which Itchy thinks is worth nearly 10 units.

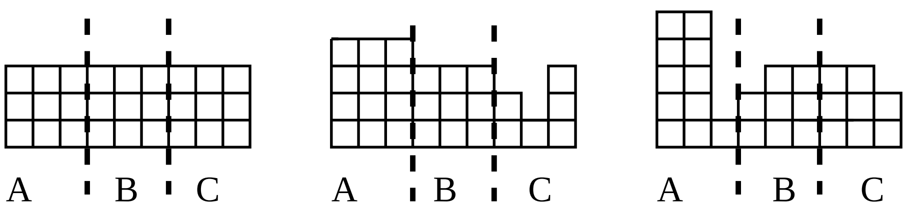
8. (15 points) Bart, Lisa, and Maggie are dividing a cake. At Lisa's suggestion, the *lone-divider* method will be used. Bart grabs the knife and will be the lone divider. How will the cake be split between them? Bart, Lisa, and Maggie's views of the cake are as follows:



Look at your answer. Is it proportional? Explain what this means.

Is it envy-free? Explain why / why not.

Each thinks the cake is worth 27 units and demands at least $\frac{27}{3} = 9$ units as his or her fair share. Bart divides up the cake as follows:



Lisa approves of pieces A and B. Maggie approves of piece A. As they approve of two different pieces, each can be given a piece she approves of. Give Lisa piece B and Maggie piece A. Bart then gets the leftover piece, piece C.

Yes, this division is proportional. Everyone gets 9 units or more, one-third of the cake or more.

No, this division is not envy-free. From Lisa's point of view, Maggie has received 12 units of cake while Lisa has only received 9 units of cake. Lisa envies Maggie.