

Logarithmic Differentiation

- When do you use logarithmic differentiation?

You use logarithmic differentiation when you have expressions of the form $y = f(x)^{g(x)}$, a variable to the power of a variable. The power rule and the exponential rule do not apply here.

- Why do you use logarithmic differentiation?

The advantage is that you can now write this as $\ln y = g(x) \ln f(x)$ – this can now be differentiated using the chain rule on the left and the product and chain rules on the right.

$$\begin{aligned}\frac{1}{y} \frac{dy}{dx} &= g(x) \frac{1}{f(x)} f'(x) + g'(x) \ln f(x) \\ \frac{dy}{dx} &= y \left(g(x) \frac{1}{f(x)} f'(x) + g'(x) \ln f(x) \right) \\ \frac{dy}{dx} &= f(x)^{g(x)} \left(g(x) \frac{1}{f(x)} f'(x) + g'(x) \ln f(x) \right)\end{aligned}$$

You want to find $\frac{dy}{dx}$ (not $\frac{1}{y} \frac{dy}{dx}$), so multiply across by $y = f(x)^{g(x)}$. Your answer should feature only x 's. You know what y is equal to; use that information.

- Look at this example $y = x^{e^x}$.

$$\begin{aligned}\ln y &= e^x \ln x \\ \frac{1}{y} \frac{dy}{dx} &= e^x \frac{1}{x} + e^x \ln x \\ \frac{dy}{dx} &= x^{e^x} \left(e^x \frac{1}{x} + e^x \ln x \right)\end{aligned}$$

- You could also use logarithmic differentiation to simplify some other expressions:

$$\begin{aligned}y &= \sqrt{x+1} \sqrt[3]{x+2} \sqrt[4]{x+3} \\ \log y &= \frac{1}{2} \log(x+1) + \frac{1}{3} \log(x+2) + \frac{1}{4} \log(x+3),\end{aligned}$$

where the second expression is easier to differentiate than the first.

Implicit Differentiation

An expression like $y = x^2 + 3$ defines the function y *explicitly* – it says openly and directly how to find y if you know x . If I have an expression like $x = \arctan(x+y)$, the connection between x and y is still there but only *implicitly*, the link is hidden and the x s and y s are tangled together.

- When do you use implicit differentiation?

You use implicit differentiation when the function y is only implicitly defined. When it is difficult (or impossible) to solve and get $y = \dots$, we use implicit differentiation.

- How do you use implicit differentiation?

You differentiate both sides of the equation and then solve for $\frac{dy}{dx}$. It is important to remember that y is a function of x and you have to use the product, quotient and chain rules just as you would for any other function of x .

- An example:

$$\begin{aligned}
 xy &= x^2 + y^2 \\
 1 \cdot y + x \frac{dy}{dx} &= 2x + 2y \frac{dy}{dx} \\
 y - 2x &= (2y - x) \frac{dy}{dx} \\
 \frac{y - 2x}{2y - x} &= \frac{dy}{dx}
 \end{aligned}$$

xy is a product of two functions of x so we use the product rule. The derivative of y is $\frac{dy}{dx}$, which you can also write y' (but make sure not to confuse this with y^1 , which has a very different meaning). On the right, we use the chain rule to differentiate y^2 . Seeing as we do not have a neat way of writing y in terms of x , it is fine if your answer includes ys .

- Finding $\frac{d^2y}{dx^2}$.

First we find $\frac{dy}{dx}$ and we differentiate again. Remember that y is still a function of x and must be treated in the same way as above. Finally, our answer for $\frac{d^2y}{dx^2}$ should contain only x s and y s and not $\frac{dy}{dx}$. Anywhere we have a $\frac{dy}{dx}$, we should fill in our answer from above.

Returning to the example above, we use the quotient rule to get

$$\begin{aligned}
 \frac{(2y - x)\left(\frac{dy}{dx} - 2\right) - (y - 2x)\left(2\frac{dy}{dx} - 1\right)}{(2y - x)^2} &= \frac{d^2y}{dx^2} \\
 \frac{(2y - x)\left(\frac{y-2x}{2y-x} - 2\right) - (y - 2x)\left(2\frac{y-2x}{2y-x} - 1\right)}{(2y - x)^2} &= \frac{d^2y}{dx^2}.
 \end{aligned}$$