Math 220 AD8&AD9 Practice Exam

This practice exam is intended to give you a chance to sit a midterm exam for Calc 1 in a classroom setting under time pressure without affecting your grade.

Disclaimer: The presence or absence of specific material on this practice exam is no guarantee of the presence or absence of that material on the actual midterm exam. All material covered in the course so far is fair game for the exam.

No calculators are permitted. Cell phones, MP3 players, and other recreational electronic devices must be turned off and put away.

1. (a) Given a function $f(x)$, define what it means for f to be *one to one.*

A function f is called one-to-one when for every $y \in \text{Range}(f)$, there is exactly one $x \in \text{Domain}(f)$ for which $y = f(x)$.

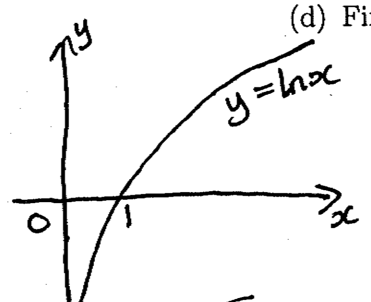
(b) Is $f(x) = x^2$ with domain $\mathbb{R}$ invertible? Explain your answer.

No, f is not invertible. It is not one-to-one and so cannot have an inverse function.
For example, $f(2) = 2^2 = 4 = (-2)^2 = f(-2)$, so we cannot define $f^{-1}(4)$.

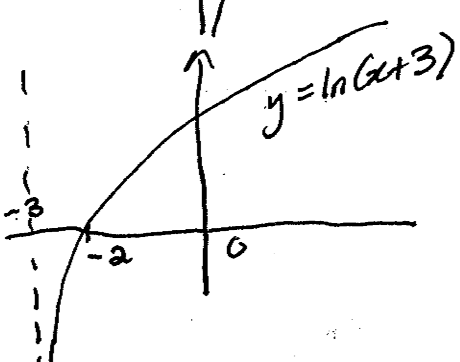
(c) Find the domain of the function $f(x) = \ln(x+3)$.

For $\ln(x+3)$ to make sense, we require
$x+3 > 0$
Domain $x > -3$, or $(-3, +\infty)$.

(d) Find the range of the function $f(x) = \ln(x+3)$.



$\ln x$ has range $(-\infty, \infty)$
and so will $f(x) = \ln(x+3)$
(It has the same graph, shifted 3 to the right.)



2. Evaluate the following limits if they exist, or explain why they do not exist. Show all of your work and use proper notation.

(a) $\lim_{x\to 5} \frac{1}{x^2(x+5)}$ This function is continuous at $x=5$ and so the limit

equals $\frac{1}{5^2(5+5)} = \frac{1}{25\times 10} = \frac{1}{250}$.

(b) $\lim_{x\to -1} \frac{x^2-2x-3}{x^2-5x-6}$ Not continuous / defined at $x=-1$.

Try factoring the numerator and denominator.

$= \lim_{x\to -1} \frac{(x+1)(x-3)}{(x+1)(x-6)} = \lim_{x\to -1} \frac{x-3}{x-6} = \frac{1-3}{1-6} = \frac{-2}{-5} = \frac{2}{5}$

(c) $\lim_{x\to 16} \frac{4-\sqrt{x}}{16-x}$ Not continuous/defined at $x=16$.

Multiply above and below by conjugate to get rid of $\sqrt{\cdot}$.

$= \lim_{x\to 16} \frac{4-\sqrt{x}}{16-x} \cdot \frac{4+\sqrt{x}}{4+\sqrt{x}} = \lim_{x\to 16} \frac{16-x}{16-x} \cdot \frac{1}{4+\sqrt{x}} = \lim_{x\to 16} \frac{1}{4+\sqrt{x}} = \frac{1}{4+\sqrt{16}}$

$= \frac{1}{4+4} = \frac{1}{8}$. (Usually only need to multiply out two conjugates and not other terms. Here, for example, easier if you do not multiply out in denominator.)

(d) $\lim_{x\to 1} \frac{1}{(x-1)^2}$

Clearly, no further simplification possible.

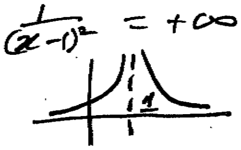
$\lim_{x\to 1^+} \frac{1}{(x-1)^2} = +\infty$. (If $x>1$, $(x-1)^2>0$. If x is very close to 1, $(x-1)^2$ will be a very small positive number and $\frac{1}{(x-1)^2}$ a very large positive number)

$\lim_{x\to 1^-} \frac{1}{(x-1)^2} = +\infty$ (Similar reason)

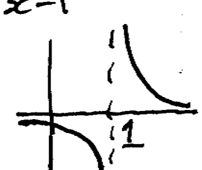
Left- and right-hand limits agree so

$\lim_{x\to 1} \frac{1}{(x-1)^2} = +\infty$.

$\lim_{x\to 1} \frac{1}{(x-1)^2} = +\infty$



but $\lim_{x\to 1} \frac{1}{x-1}$ DNE



3. A velociraptor's position after t minutes of running in a straight line is given by $p(t) = 9t^2 - t^3$, $t \in [0, 3]$. Find the velociraptor's average velocity over this time interval.

Aver. Velocity = $\frac{\text{Displacement}}{\text{Time}}$ Time elapsed = $3 - 0 = 3$.

Displacement = $p(3) - p(0) = (9(3)^2 - 3^3) - (9(0)^2 - 0^3)$
$= 81 - 27 = 54$.

Average Velocity = $\frac{54}{3} = 18$ (units of length per minute)

4. Use the limit definition of the derivative to find $f'(x)$ for $f(x) = x^2 - 7x$. (No credit would be given for using other methods.)

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^2 - 7(x+h) - [x^2 - 7x]}{h}$$

$$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - 7x - 7h - x^2 + 7x}{h}$$

$$= \lim_{h \to 0} \frac{2xh + h^2 - 7h}{h}$$

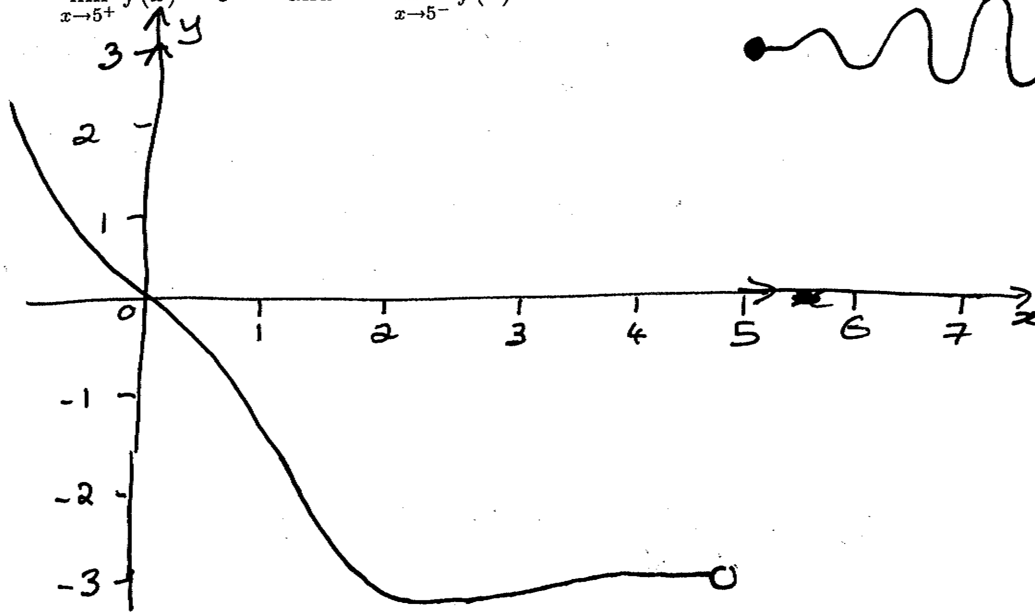
$$= \lim_{h \to 0} 2x + h - 7$$

$$= 2x - 7.$$

$$f'(x) = 2x - 7$$

5. (a) Give a graphical example of a function $f(x)$ satisfying

$$\lim_{x \to 5^+} f(x) = 3 \qquad \text{and} \qquad \lim_{x \to 5^-} f(x) = -3.$$



(b) Is the function you drew continuous at $x = 5$? Explain your answer clearly.

This function is not continuous at $x=5$ because $\lim_{x\to 5^+} f(x) \neq \lim_{x\to 5^-} f(x)$ and thus $\lim_{x\to 5} f(x)$ does not exist.

3 ~~pa~~ parts to definition of continuity (at $x=a$)

- f is defined at $x=a$.
- $\lim_{x\to a} f(x)$ exists.
- $f(a) = \lim_{x\to a} f(x)$.

6. Identify and classify all discontinuities of the following function. Justify your answers using limits.

$$f(x) = \begin{cases} \frac{x^2-4}{x-2} & \text{for } x \geq 1, \\ \frac{1}{x} & \text{for } x < 1. \end{cases}$$

Clearly should examine the points $x=0$, $x=1$, $x=2$ more carefully. Function is not defined at $x=0$ and $x=2$, so discontinuous at $x=0$ and $x=2$.

$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} \frac{(x-2)(x+2)}{x-2} = \lim_{x\to 2} x+2 = 4.$

$\lim_{x\to 2} f(x)$ exists so there is a removable discontinuity at $x=2$.

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{x^2-4}{x-2} = \frac{1-4}{1-2} = \frac{-3}{-1} = 3.$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{1}{x} = \frac{1}{-1} = -1.$

$\Big\}$ $\lim_{x\to 1^+} f(x) \neq \lim_{x\to 1^-} f(x)$

$\lim_{x\to 1} f(x)$ does not exist, so f is discontinuous at $x=1$. ("Jump discontinuity")

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{1}{x} = +\infty$. $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{1}{x} = -\infty$. Vertical asymptote at $x=0$

7. Use the Intermediate Value Theorem to prove that the function $f(x) = x^7 - 7x^2 + 5$ has a zero in the interval $[0, 1]$.

$f(0) = 5.$

$f(1) = 1 - 7 + 5 = -1.$

$f(0) > 0$, $f(1) < 0$

so by the intermediate value theorem, there must exist $x \in [0,1]$ such that $f(x) = 0$.