Math 241, Spring 2007, Merit Worksheet 4

1. A particle has position vector $\mathbf{r}(t) =< 2 - t, 1 + t^2 >$. Sketch the path traced by this particle as t increases from -2 to 2. Label the points $\mathbf{r}(-2), \mathbf{r}(-1), \mathbf{r}(0), \mathbf{r}(1), \mathbf{r}(2)$. Find an equation for this curve using cartesian coordinates.

2. Which of the following is not a parametrization of the entire curve $y = x^3$?

   a. $x(t) = t; \quad y(t) = t^3$, **b**. $x(t) = t^2; \quad y(t) = t^6$,
   c. $x(t) = t^3; \quad y(t) = t^9$, **d**. $x(t) = 2t; \quad y(t) = 8t^3$.

   What are the differences between those that do parameterize the curve $y = x^3$?

3. Find the values of $\mathbf{r}'(1), \mathbf{r}''(1)$ and $v(1)$ for

$$\mathbf{r}(t) =< t^3, 2t^2 + 1 >$$

4. A point moves with constant speed, so its velocity vector $\mathbf{v}$ satisfies the condition

$$|\mathbf{v}|^2 = \mathbf{v} \cdot \mathbf{v} = C(\text{ a constant}).$$

   Prove that the velocity and acceleration vectors of the point are always perpendicular to each other.

5. Give an example of a curve in $\mathbb{R}^2$ or $\mathbb{R}^3$ for which the position at time $t = 10$ is the same as the position at time $t = 0$ but for which the speed is never 0.

   Can you find a differentiable function of a single variable with this property, i.e., a function $y = f(x)$ for which $f(10) = f(0)$ and $f'(x) \neq 0$ for $x \in [0, 10]$? Why or why not?

6. Find a parameterization of the curve formed by the intersection of the two surfaces:

$$x^2 + y^2 = 4 \qquad \text{and} \qquad x + y + z = 1$$

7. Find the parametric and symmetric equations for the tangent line to $\mathbf{r}(t) =< t^2, -2t, 1 + t^3 >$ at the point $(1, 2, 0)$.

   Find a tangent vector of length 2 at $(4, -4, 9)$.

8. Jethro throws a mango off a cliff and into the sea below. The cliff is 200m high and the mango is thrown with initial velocity $< 20, 30 >$ (measured in m/s). What is the greatest height the mango reaches? How much time passes before the fruit hits the waves? If Jethro changes his mind and decides to eat the mango, how far out will he have to swim to retrieve the floating mango? (Ignore air resistance but don't ignore gravity.)

9. Consider the curve $\mathbf{r}(t) = \sin t\mathbf{i} + t\mathbf{j} + \cos t\mathbf{k}$, $0 \leq t \leq 2\pi$.

   (a) Find the arc length of $\mathbf{r}(t)$.

   (b) Find the arc length of $\mathbf{s}(t) = \sin t^2\mathbf{i} + t^2\mathbf{j} + \cos t^2\mathbf{k}$, $0 \leq t \leq \sqrt{2\pi}$.

   (c) Is there a relationship between $\mathbf{r}$ and $\mathbf{s}$?

   (d) What does this tell us about arc length?

10. Find the arc-length parametrization of the helix

$$x(t) = 3\cos t, \quad y(t) = 3\sin t, \quad z(t) = 4t$$

    in terms of the arc length s measured from the initial point $(3, 0, 0)$.

11. What is the curvature at the points marked in the diagram?

12. Using the example of Problem 1, discuss the following terms: radius of curvature of a function, center of curvature, osculating circle (or circle of curvature). Find all of these for the point $(1, 1)$ on the curve $xy = 1$.

Warm-Up Problems for Next Time

1. There are a lot of formulae in Section 11.6. You will need to memorise these. Write out the formulae for arc-length, curvature, tangential and normal components of acceleration. How do you decide what formula to use for curvature?