Math 241, Spring 2007, Merit Worksheet 5

1. Consider the curve $\mathbf{r}(t) = \sin t\mathbf{i} + t\mathbf{j} + \cos t\mathbf{k}$, $0 \leq t \leq 2\pi$.

   (a) Find the arc length of $\mathbf{r}(t)$.

   (b) Find the arc length of $\mathbf{s}(t) = \sin t^2\mathbf{i} + t^2\mathbf{j} + \cos t^2\mathbf{k}$, $0 \leq t \leq \sqrt{2\pi}$.

   (c) Find the arc length of $\mathbf{w}(f) = \cos t\mathbf{i} + (t + \pi/2)\mathbf{j} - \sin t\mathbf{k}$, $-\pi/2 \leq t \leq 3\pi/2$.

   (d) Is there a relationship between $\mathbf{r}, \mathbf{s}$ and $\mathbf{w}$?

   (e) What does this tell us about arc length?

2. Find the arc-length parametrization of the helix

$$x(t) = 3\cos t, \quad y(t) = 3\sin t, \quad z(t) = 4t$$

   in terms of the arc length s measured from the initial point $(3, 0, 0)$.

3. What is the curvature at the points marked in the diagram?

4. Using the example of the previous problem, discuss the following terms: radius of curvature of a function, center of curvature, osculating circle (or circle of curvature). Find all of these for the point $(1, 1)$ on the curve $xy = 1$.

5. Consider the curve $\gamma(t) =< e^{-t}\cos t, e^{-t}\sin t, e^{-t} >$, $t \geq 0$.

   (a) Graph/describe the curve and prove that the curve lies on the surface $z^2 = x^2 + y^2$.

   (b) Find the curvature.

   (c) What happens to the curvature as $t \to \infty$?

   (d) Prove that as $t \to \infty$, the total distance travelled is $\sqrt{3}$.

6. Find the curvature κ, the unit tangent vector $\mathbf{T}$, the unit normal vector $\mathbf{N}$, the tangential and normal components of acceleration for the space curve with position vector $\gamma(t) =< t, t^2, t^3 >$ at the point $(1, 1, 1)$.

7. Ask me for a piece of aluminum foil. What directions are the unit tangent, unit normal vectors pointing in? What is the osculating plane? The osculating circle? Where is the curvature least? greatest?

8. Using tangential and normal components of acceleration, describe in words those curves $\mathbf{r}(t)$ for which

   (a) $\mathbf{a} \perp \mathbf{v}$

   (b) $\mathbf{a} \,||\mathbf{v}$

      i. Assuming initial position r_0 and initial velocity v_0, find an equation for the velocity of $\mathbf{v}$.

      ii. Find an equation for the position $\mathbf{r}(t)$.

      iii. Describe the line which contains the range of the position vector $\mathbf{r}(t)$.

Warm-Up Problems for Next Time

1. Convert to both cylindrical and spherical coordinates $(1, 1, 1)$.