MATH 225 February 21, 2002

Name: key

Exam 1

Justify all your work. Partial credit will be given if you show your reasoning. Each problem is worth 10 *points unless otherwise indicated.*

(1) Let A be an $n \times n$ matrix. List 3 different statements that are equivalent to the statement "A is an invertible matrix."

The following responses or their equivalent were acceptable:

Solution:

(a) *A is row equivalent to the $n \times n$ identity matrix.*
(b) *A has n pivot positions.*
(c) *The equation $A\vec{x} = \vec{0}$ has only the trivial solution.*
(d) *The columns of A form a linearly independent set.*
(e) *The equation $A\vec{x} = \vec{0}$ has at least one solution for every $\vec{b}$ in $\mathbb{R}^n$.*
(f) *The columns of A span $\mathbb{R}^n$.*
(g) *There is an $n \times n$ matrix C such that $CA = I$.*
(h) *There is an $n \times n$ matrix D such that $AD = I$*
(i) *A^T is an invertible matrix.*

(2) Use only definitions and simplest properties of matrix multiplication to show that if A is invertible and $AC = I$, then $C = A^{-1}$.

Solution:

$$\begin{aligned} AC = I &\implies AA^{-1}C = A^{-1}I \\ &\implies IC = A^{-1} \\ &\implies C = A^{-1}. \end{aligned}$$

(3) Find the inverse of $\begin{bmatrix} 1 & 0 & 5 \\ 1 & 1 & 0 \\ 3 & 2 & 6 \end{bmatrix}$. Show your work.

Solution:

$$\begin{bmatrix} 1 & 0 & 5 \\ 1 & 1 & 0 \\ 3 & 2 & 6 \end{bmatrix}^{-1} = \begin{bmatrix} 6 & 10 & -5 \\ -6 & -9 & 5 \\ -1 & -2 & 1 \end{bmatrix}.$$

(4) Solve the system

$$\begin{cases} x_1 & & + & 5x_3 & = & 1 \\ x_1 & + & x_2 & & = & 2 \\ 3x_1 & + & 2x_2 & + & 6x_3 & = & 1 \end{cases}.$$

[*Note: your answer to problem 3 might be useful.*]

Solution:

Since you want to solve the matrix equation

$$\begin{bmatrix} 1 & 0 & 5 \\ 1 & 1 & 0 \\ 3 & 2 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix},$$

you need to multiply both sides of the equation on the left by A^{-1} *above. In other words*

$$\begin{aligned} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 5 \\ 1 & 1 & 0 \\ 3 & 2 & 6 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 6 & 10 & -5 \\ -6 & -9 & 5 \\ -1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 21 \\ -19 \\ -4 \end{bmatrix}. \end{aligned}$$

(5) Which of the following statements are false, in general? Read the statements carefully. (20 points)

(a) If A and B are 3×3 matrices and $B = [\vec{b_1}\ \vec{b_2}\ \vec{b_3}]$, then $AB = [A\vec{b_1} + A\vec{b_2} + \vec{b_3}]$.
(b) The columns of a matrix A are linearly dependent if the equation $A\vec{x} = \vec{0}$ has only the trivial solution.
(c) The second row of AB is the second row of A multiplied on the right by B.
(d) $(AB)C = (AC)B$.
(e) $(AB)^T = A^T B^T$.
(f) The transpose of a sum of matrices equals the sum of their transposes.
(g) If a set $\{\vec{v_1}, \vec{v_2}, \dots, \vec{v_n}\}$ contains the zero vector, then the set is linearly independent.
(h) A homogeneous equation is always consistent.
(i) The solution set of $A\vec{x} = \vec{b}$ is obtained by translating the solution set of $A\vec{x} = \vec{0}$
(j) If the equation $A\vec{x} = \vec{b}$ has a solution then $\vec{b}$ is a linear combination of the columns of A.

Solution:

The false statements are

$$(a), (b), (d), (e), \textit{and}\, (g), .$$

(6) If $C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, find C^3.

Solution:

$$\begin{aligned} C^3 &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^3 \\ &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 37 & 54 \\ 81 & 118 \end{bmatrix} \end{aligned}$$

(7) Decide if the vectors

$$\vec{v_1} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \vec{v_2} = \begin{bmatrix} -3 \\ -5 \\ 6 \end{bmatrix}, \vec{v_3} = \begin{bmatrix} 0 \\ 5 \\ -6 \end{bmatrix}.$$

are linearly independent. Justify your answer. You may assume that

$$\begin{bmatrix} 1 & -3 & 0 \\ 3 & -5 & 5 \\ -2 & 6 & -6 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 3/4 & 5/8 \\ -1/3 & 1/4 & 5/24 \\ -1/3 & 0 & -1/6 \end{bmatrix}.$$

Solution:
The three vectors above are linearly independent if and only if the homogeneous matrix equation

$$\begin{bmatrix} 1 & -3 & 0 \\ 3 & -5 & 5 \\ -2 & 6 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

has only the trivial solution

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

There are a number of ways to show this, some of which use the Invertible Matrix Theorem. One acceptable answer is that

$$\begin{bmatrix} 1 & -3 & 0 \\ 3 & -5 & 5 \\ -2 & 6 & -6 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 3/4 & 5/8 \\ -1/3 & 1/4 & 5/24 \\ -1/3 & 0 & -1/6 \end{bmatrix},$$

so by the Invertible Matrix Theorem the above system has only the trivial solution. Alternatively, you could row reduce the appropriate augmented matrix.

Additionally, you could also row reduce

$$\begin{bmatrix} 1 & -3 & 0 \\ 3 & -5 & 5 \\ -2 & 6 & -6 \end{bmatrix}$$

until it became apparent that there are three pivot positions. Then, using the Invertible Matrix Theorem, you could conclude the homogeneous equation above has only the trivial solution, so the vectors are linearly independent.

(8) In problem 7, is $\vec{v_3}$ in the span of $\vec{v_1}$ and $\vec{v_2}$? You may assume that

$$\begin{bmatrix} 1 & -3 & 0 \\ 3 & -5 & 5 \\ -2 & 6 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 0 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}.$$

Solution:

The question is whether or not we can find real numbers x_1 and x_2 such that

$$x_1 \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ -5 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 6 \end{bmatrix}.$$

That is, we want to solve the matrix equation

$$\begin{bmatrix} 1 & -3 \\ 3 & -5 \\ -2 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 6 \end{bmatrix}.$$

To this end, we form the augmented matrix

$$\begin{bmatrix} 1 & -3 & 0 \\ 3 & -5 & 5 \\ -2 & 6 & 6 \end{bmatrix},$$

and find

$$\begin{bmatrix} 1 & -3 & 0 \\ 3 & -5 & 5 \\ -2 & 6 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 0 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}.$$

Hence, the system is inconsistent, so we cannot find the necessary real numbers x_1 and x_2 to express $\vec{v_3}$ as a linear combination of $\vec{v_1}$ and $\vec{v_2}$. Thus, $\vec{v_3}$ is not in the span of $\vec{v_1}$ and $\vec{v_2}$.

(9) Consider the production model $\vec{x} = C\vec{x} + \vec{d}$ for an economy with two sectors, where

$$C = \begin{bmatrix} .1 & .6 \\ .5 & .2 \end{bmatrix}, \qquad \vec{d} = \begin{bmatrix} 18 \\ 11 \end{bmatrix}.$$

Use an inverse matrix to determine the production level, $\vec{x}$, necessary to satisfy the final demand. You may assume that

$$\begin{bmatrix} .9 & -.6 \\ -.5 & .8 \end{bmatrix}^{-1} = \begin{bmatrix} 1.9 & 1.4 \\ 1.2 & 2.1 \end{bmatrix}.$$

Solution:

Note the Leonteif Input-Output Model:

$$\begin{array}{ccccc} \textit{(amount produced)} & = & \textit{(intermediate demand)} & + & \textit{(Final Demand)} \\ x & = & Cx & + & d \end{array}.$$

Recall that using matrix algebra, this leads to the equation

$$x = (I - C)^{-1} d.$$

In this case,

$$I - C = \begin{bmatrix} .9 & -.6 \\ -.5 & .8 \end{bmatrix},$$

so

$$(I - C)^{-1} = \begin{bmatrix} 1.9 & 1.4 \\ 1.2 & 2.1 \end{bmatrix}.$$

Finally,

$$(I - C)^{-1} d = \begin{bmatrix} 1.9 & 1.4 \\ 1.2 & 2.1 \end{bmatrix} \begin{bmatrix} 18 \\ 11 \end{bmatrix} = \begin{bmatrix} 49.6 \\ 44.7 \end{bmatrix}.$$