MATH 225 April 2, 2002

Name: ____________________________

Exam 2

Justify all your work. Partial credit will be given if you show your reasoning. Each problem is worth 10 points unless otherwise indicated.

(1) Solve the equation

$$\begin{vmatrix} -\lambda & 5 \\ 2 & 3-\lambda \end{vmatrix} = 0$$

for λ.

Taking the determinant on the left side of the equations yields

$$\begin{aligned} -\lambda(3-\lambda) - 2\cdot 5 &= 0 \\ \lambda^2 - 3\lambda - 10 &= 0 \\ (\lambda-5)(\lambda+2) &= 0 \end{aligned}$$

Thus,

$$\lambda \in \{5, -2\}.$$

(2) let A be a 3×3 matrix with $\det A = 2$.

(a) Find $\det(A^2)$.

$$\det A^2 = \det A \det A = 2\cdot 2 = 4.$$

(b) Find $\det(A^k)$.

$$\det A^k = \underbrace{\det A \det A \ \cdots\ \det A}_{k \text{ times}} = 2^k.$$

(c) Find $\det(A^T)$.

$$\det A^T = \det A = 2.$$

(d) Find $\det(A^{-1})$.

$$\det A^{-1} = \frac{1}{\det A} = \frac{1}{2}.$$

(e) Find $\det(AA^{-1})$.

$$\det(AA^{-1}) = \det(I) = 1,$$

or

$$\det(AA^{-1}) = \det A \det A^{-1} = \frac{\det A}{\det A} = 1.$$

(3) Calculate the area of the triangle determined by the points $(0,0)$, $(2,5)$, $(6,1)$.

Hint: *The area of a triangle formed by connecting the points defined by* $\vec{0}, \vec{v_1}$, *and* $\vec{v_2}$ *in* $\mathbb{R}^2$ *is half the area of the parallelogram determined by* $\vec{v_1}, \vec{v_2}$.

Let

$$A = \begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix}.$$

The area of the parallelogram determined by the points $(2,5)$ and $(6,1)$ is $|\det A|$. Thus, the area of the triangle determined by $(2,5)$, $(6,1)$ and $(0,0)$ is

$$\frac{1}{2}\begin{vmatrix} 6 & 2 \\ 1 & 5 \end{vmatrix} = \frac{1}{2}(6 \cdot 5 - 2 \cdot 1) = \frac{1}{2} \cdot 28 = 14.$$

(4) Let

$$W = \left\{ \begin{bmatrix} a - 5b \\ 2a \\ 2b + 1 \\ b \end{bmatrix} : a, b \in \mathbb{R} \right\}.$$

Is W a subspace of $\mathbb{R}^4$? Why or why not?

W is not a subspace of $\mathbb{R}^4$ since W does not contain the zero vector $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. *To see this, note that obtaining the zero vector in W would require us to find $a, b \in \mathbb{R}$ so that simultaneously, $2a = 0$, $b = 0$, and $2b + 1 = 0$, an impossibility.*

(5) Let $A = \begin{bmatrix} 1 & 1 & 3 & 1 \\ 2 & 3 & 1 & 1 \\ 1 & 0 & 8 & 2 \end{bmatrix}$, and assume $A \sim \begin{bmatrix} 1 & 0 & 8 & 2 \\ 0 & 1 & -5 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Find a basis for $Nul\ A$.

From the row reduced echelon form of A, we see that the general solution to $A\vec{x} = \vec{0}$ is given by

$$\begin{cases} x_1 = -8x_3 - 2x_4 \\ x_2 = 5x_3 + x_4 \\ x_3, x_4 \text{ free} \end{cases}.$$

Thus,

$$Nul\ A = \left\{ x_3 \begin{bmatrix} -8 \\ 5 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix} : x_3, x_4 \in \mathbb{R} \right\}.$$

That is, $\vec{w}$ is in $Nul\ A$ if and only if $\vec{w}$ is a linear combination of $\begin{bmatrix} -8 \\ 5 \\ 1 \\ 0 \end{bmatrix}$ *and* $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$.

Therefore, a basis for the null space is given by

$$\mathcal{B} = \left\{ \begin{bmatrix} -8 \\ 5 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Note that $\dim \mathcal{B}$ *should be equal to the number of free variables in the general solution of* $A\vec{x} = 0$, *and our answer is consistent with this requirement.*

(6) Find a basis for the subspace of $\mathbb{R}^2$ spanned by

$$\begin{bmatrix} 2 \\ -5 \end{bmatrix}, \begin{bmatrix} -4 \\ 10 \end{bmatrix}, \begin{bmatrix} -3 \\ 6 \end{bmatrix}.$$

Let

$$V = Span \left\{ \begin{bmatrix} 2 \\ -5 \end{bmatrix}, \begin{bmatrix} -4 \\ 10 \end{bmatrix}, \begin{bmatrix} -3 \\ 6 \end{bmatrix} \right\}.$$

Our goal is to find a basis for V. *Set* $A = \begin{bmatrix} 2 & -4 \\ -5 & 10 \end{bmatrix}$ *and* $B = \begin{bmatrix} 2 & -3 \\ -5 & 6 \end{bmatrix}$. *Then*

$$\det A = 2 \cdot 10 - (-5) \cdot (-4) = 20 - 20 = 0.$$

Thus, A *is not invertible and* B ***is*** *invertible. Thus,* $\begin{bmatrix} 2 \\ -5 \end{bmatrix}$ *and* $\begin{bmatrix} -4 \\ 10 \end{bmatrix}$ *are linearly dependent (you could also see they are dependent since one is a multiple of the other). Thus, by the Spanning Theorem,* $V = Span \left\{ \begin{bmatrix} 2 \\ -5 \end{bmatrix}, \begin{bmatrix} -3 \\ 6 \end{bmatrix} \right\}$. *Since* $\det B \neq 0$, *the columns of* B *form a linearly independent set*[1], *and they span* V. *Then, by definition of span, the columns of* B,

$$\left\{ \begin{bmatrix} 2 \\ -5 \end{bmatrix}, \begin{bmatrix} -3 \\ 6 \end{bmatrix} \right\}$$

is a basis for V.

There are other valid solutions. For example, since the pivot columns of $\begin{bmatrix} 2 & -4 & -3 \\ -5 & 10 & 6 \end{bmatrix}$ *form a basis for* $Col\ A = V$, *showing*

$$\begin{bmatrix} 2 & -4 & -3 \\ -5 & 10 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & -\frac{15}{2} \end{bmatrix}$$

would allow you to conclude that a basis for V *is given by*

$$\left\{ \begin{bmatrix} 2 \\ -5 \end{bmatrix}, \begin{bmatrix} -3 \\ 6 \end{bmatrix} \right\}.$$

[1] Similar arguments could be made to show that

$$\left\{ \begin{bmatrix} -4 \\ 10 \end{bmatrix}, \begin{bmatrix} -3 \\ 6 \end{bmatrix} \right\}$$

is a basis for B.

(7) Let A be a $n \times n$ matrix. Suppose that $Nul\ A = \{\vec{0}\}$.

(a) Do the columns of A form a basis for $\mathbb{R}^n$? Why or why not? [**Hint:** *Think about what the null space of A is.*]

Since the null space of A is $\{\vec{0}\}$, the only solution to the equation $A\vec{x} = \vec{0}$ is the trivial solution. By the invertible matrix theorem, A is invertible and the columns of A form a basis for $\mathbb{R}^n$.

(b) Can $\det A = 0$? Explain your answer.

No, A is invertible if and only if $\det A \neq 0$. Since we know that any A with $Nul\ A = \{\vec{0}\}$ is invertible, we know the determinant of any such A is nonzero.

(8) Let A be an $m \times n$ matrix. If $Rank\ A = n$, can $m < n$ be true? Explain your answer. [**Remark:** *If $m < n$ is possible, an example where this occurs would suffice. If $m < n$ can never happen, explain why.*]

The rank of A is the dimension of the column space. Thus, there are n vectors in any basis for $Col\ A$. Since the pivot columns of A form a basis for $Col\ A$, we must have n pivot columns in A. However, if $m < n$, there arer more column vectors in A then there are entries in each column. Thus, if we assume that $Rank\ A = n$, we can conclude that it is never possible for $m < n$ to be true.

(9) Let $W = \{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}\}$ be vectors in a vector space V. What **two** properties must the vectors in W satisfy so that W is a basis for V?

(a) *The vectors in W must form a linearly independent set.*
(b) *The span of the vectors in W must equal V.*

(10) Let $A = \begin{bmatrix} 5 & -18 & 69 \\ 10 & -41 & 178 \\ 1 & -4 & 17 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 3 \\ 23 \\ 2 \end{bmatrix}$. You may assume

$$\begin{bmatrix} 5 & -18 & 69 & 3 \\ 10 & -41 & 178 & 23 \\ 1 & -4 & 17 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 1 & -5 \\ 0 & -1 & 8 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(a) Determine if $\vec{b}$ is in the column space of A.

We know that $\vec{b}$ is in the column space of A if and only if there is a solution to the equation $A\vec{x} = \vec{b}$. Forming the associated augmented matrix $\begin{bmatrix} 5 & -18 & 69 \\ 10 & -41 & 178 \\ 1 & -4 & 17 \end{bmatrix}$ *and noting* $\begin{bmatrix} 1 & -2 & 1 & -5 \\ 0 & -1 & 8 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ *is a row equivalent matrix, we see that $A\vec{x} = \vec{b}$ is* *inconsistent. Since, for this* $\vec{b}$*, there is no solution to* $A\vec{x} = \vec{b}$*, we can conclude that* $\vec{b}$ *is not in the column space of* A*.*

(b) Find a basis for the column space of A.

The pivot columns of A *form a basis for Col* A*. From the given information, we can conclude that the first and second columns of* A *are pivot columns. Hence, a basis for the column space of* A *is given by*

$$\left\{ \begin{bmatrix} 5 \\ 10 \\ 1 \end{bmatrix}, \begin{bmatrix} -18 \\ -41 \\ -4 \end{bmatrix} \right\}.$$