Question: Find a matrix A and vectors $\vec{b}$ for which $A\vec{x} = \vec{b}$ has at least one solution, yet it is not true that $A\vec{x} = \vec{b}$ has at least one solution for all $\vec{b}$ in $\mathbb{R}^n$.

Consider the coefficient matrix $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ *and the vector* $\vec{b} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$.
The augmented matrix corresponding to this system is

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since x_3 *is a free variable, this system has more than one solution. In fact, if* $\vec{b}$ *is any vector in* $\mathbb{R}^3$ *such that the third entry is* 0*, the matrix equation* $A\vec{x} = \vec{b}$ *will have infinitely many solutions.*

However, the above system does not have at least one solution for every $\vec{b}$ *in* $\mathbb{R}^3$*. Consider* $\vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$*. The corresponding augmented matrix will be*

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix},$$

which is inconsistent. In fact, any $\vec{b}$ *in* $\mathbb{R}^3$ *for which the third entry is nonzero will yield an inconsistent system* $A\vec{x} = \vec{b}$*. The statement that* $A\vec{x} = \vec{b}$ *has at least one solution for every* $\vec{b}$ *in* $\mathbb{R}^3$ *is false since for some* $\vec{b}$ *in* $\mathbb{R}^3$*, the matrix equation has no solution.*