MATH 225 February 19, 2002

Name: key

Quiz 8

Justify all your work. Partial credit will be given if you show your reasoning.

(1) If the equation $A\vec{x} = \vec{b}$ has more than one solution for some $\vec{b}$ in $\mathbb{R}^n$, do the columns of A span $\mathbb{R}^n$? Why or why not?

No.

Consider the coefficient matrix $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ *and the vector* $\vec{b} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$.

The augmented matrix corresponding to this system is

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since x_3 is a free variable, this system has more than one solution. In fact, if $\vec{b}$ is any vector in $\mathbb{R}^3$ such that the third entry is 0, the matrix equation $A\vec{x} = \vec{b}$ will have infinitely many solutions.

However, the above system does not have at least one solution for every $\vec{b}$ in $\mathbb{R}^3$. Consider $\vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. *The corresponding augmented matrix will be*

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix},$$

which is inconsistent. In fact, any $\vec{b}$ in $\mathbb{R}^3$ for which the third entry is nonzero will yield an inconsistent system $A\vec{x} = \vec{b}$. The statement that $A\vec{x} = \vec{b}$ has at least one solution for every $\vec{b}$ in $\mathbb{R}^3$ is false since for some $\vec{b}$ in $\mathbb{R}^3$, the matrix equation has no solution. The Invertible Matrix Theorem implies the columns of A do not span $\mathbb{R}^n$.

(2) Let A be an $n \times n$ matrix. List four different statements that are equivalent to the statement "A is an invertible matrix."

The following responses or their equivalent were acceptable:

Solution:

(a) *A is row equivalent to the $n \times n$ identity matrix.*

(b) *A has n pivot positions.*

(c) *The equation* $A\vec{x} = \vec{0}$ *has only the trivial solution.*
(d) *The columns of* A *form a linearly independent set.*
(e) *The equation* $A\vec{x} = \vec{0}$ *has at least one solution for every* $\vec{b}$ *in* $\mathbb{R}^n$.
(f) *The columns of* A *span* $\mathbb{R}^n$.
(g) *There is an* $n \times n$ *matrix* C *such that* $CA = I$.
(h) *There is an* $n \times n$ *matrix* D *such that* $AD = I$
(i) A^T *is an invertible matrix.*