- A system of equations has either
  (1) no solution, or
  (2) exactly one solution, or
  (3) infinitely many solutions
- **Elementary Row Operations**
  (1) (Replacement) Replace one row by the sum of itself and a multiple of another row.
  (2) (Interchange) Interchange two rows.
  (3) (Scaling) Multiply all entries in a row by a nonzero constant.
- If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set.
- Each matrix is row equivalent to one and only one reduced echelon matrix.
- A linear system is consistent if and only if the rightmost column of the augmented matrix is not a pivot column, that is, if and only if an echelon form of the augmented matrix has no row of the form

$$[0 \ \cdots 0 \ b] \qquad \text{with } b \neq 0.$$

  If a linear system is consistent then the solution set contains either a unique solution (when there are no free variables), or infinitely many solutions (when there is at least one free variable).
- If $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^2$ are represented as points in the plane, then $u + v$ corresponds to the fourth vertex of the parallelogram whose other vertices are $0, \vec{u}, \vec{v}$.
- A vector equation $x_1\vec{v_1} + x_2\vec{v_2} + \cdots + x_n\vec{v_n} = \vec{b}$ has the same solution set as the linear system whose augmented matrix is

$$[\vec{a_1} \ \vec{a_2} \ \cdots \ \vec{a_n} \ \vec{b}]$$

  In particular, $\vec{b}$ can be generated by a linear combination of $\vec{a_1}, \ldots, \vec{a_n}$ if and only if there exists a solution to the above linear system.
- If A is an $m \times n$ matrix with columns $\vec{a_1}, \ldots, \vec{a_n}$, and if $\vec{b}$ is in $\mathbb{R}^m$, the matrix equation

$$A\vec{x} = \vec{b}$$

  has the same solution set as the vector equation

$$x_1\vec{v_1} + x_2\vec{v_2} + \cdots + x_n\vec{v_n} = \vec{b}$$

  which in turn has the same solution set as the system of linear equations whose augemented matrix is

$$[\vec{a_1} \ \vec{a_2} \ \cdots \ \vec{a_n} \ \vec{b}].$$

- The equation $A\vec{x} = \vec{b}$ has a solution if and only if $\vec{b}$ is a linear combination of the columns of A.
- Let A be an $m \times n$ matrix. Then the following statements are logically equivalent. That is, for a particular A, they are all true or they are all false.
  (1) For each $\vec{b}$ in $\mathbb{R}^m$ the equation $A\vec{x} = \vec{b}$ has a solution.
  (2) The columns of A span $\mathbb{R}^m$.
  (3) A has a pivot position in every row.

- The homogeneous equation $A\vec{x} = \vec{0}$ has a nontrivial solution if and only if the equation has at least one free variable.
- Suppose the equation $A\vec{x} = \vec{b}$ is consistent for some given b, and let p be a solution. Then the solution set of $A\vec{x} = \vec{b}$ is the set of all vectors of the form $w = p + v_h$, where v_h is any solution to the homogeneous equation $A\vec{x} = \vec{0}$.
- An indexed set of vectors $\{\vec{v_1}, \dots, \vec{v_n}\}$ in $\mathbb{R}^n$ is said to be linearly independent if the vector equation
$$x_1\vec{v_1} + x_2\vec{v_2} + \cdots + x_p\vec{v_p} = \vec{0}$$
has only the trivial solution. The set $\{\vec{v_1}, \dots, \vec{v_n}\}$ is said to be linearly dependent if there exist weights $c_1, \dots, c_p$, not all zero, such that
$$c_1\vec{v_1} + c_2\vec{v_2} + \cdots + c_p\vec{v_p} = \vec{0}$$
- The columns of a matrix A are linearly independent if and only if the equation $A\vec{x} = \vec{0}$ has only the trivial solution.
- A set of two vectors $\{\vec{v_1}, \vec{v_2}\}$ is linearly dependent if and only if one of the vectors is a multiple of the other.
- An indexed set $S = \{\vec{v_1}, \dots, \vec{v_n}\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others. In fact, if S is linearly dependent, and $\vec{v_1} \neq \vec{0}$, then some $\vec{v_j}$ (with $j > 1$) is a linear combination of the preceding vectors $\vec{v_1}, \dots, \vec{v_{j-1}}$.
- If a set comtains more vectors than there are entries in each vector, then the set is linearly dependent. That is, any set $\{\vec{v_1}, \dots, \vec{v_p}\}$ in $\mathbb{R}^n$ is linearly dependent if $p > n$.
- If a set $S = \{\vec{v_1}, \dots, \vec{v_n}\}$ in $\mathbb{R}^n$ contains the zero vector, then the set is linearly dependent.