- If r is a scalar and A is a matrix, then the scalar multiple rA is the matrix whose columns are r times the corresponding columns in A.
- Let A, B be matrices of the same size, and let r and s be scalars.
  (1) $A + B = B + A$
  (2) $(A + B) + C = A + (B + C)$
  (3) $A + 0 = A$
  (4) $r(A + B) = rA + rB$
  (5) $(r + s)A = rA + sA$
  (6) $r(sA) = (rs)A$
- If A is an $n \times n$ matrix, and B is an $n \times p$ matrix with columns $\vec{b_1}, \ldots, \vec{b_p}$ then the product AB is the $m \times p$ matrix whose columns are $A\vec{b_1}, \ldots, A\vec{b_p}$. That is
$$AB = A[\vec{b_1}\ \vec{b_2} \cdots A\vec{b_p}].$$
- Each column of AB is a linear combination of the columns of A using weights from each of the corresponding columns of B.
- The i^{th} row of AB is the i^{th} row of A times B.
- Let A be $m \times n$, and let B and C have sizes for which the indicated sums and products are defined.
  (1) $A(BC) = (AB)C$
  (2) $A(B + C) = AB + AC$
  (3) $(B + C)A = BA + CA$
  (4) $r(AB) = (rA)B = A(rB)$ for any scalar r
  (5) $I_m A = A = AI_n$
- Let A and B denote matrices whose sizes are appropriate for the following sums and products.
  (1) $(A^T)^T = A$
  (2) $(A + B)^T = A^T + B^T$
  (3) For any scalar r, $(rA)^T = rA^T$
  (4) $(AB)^T = B^T A^T$. In other words, the transpose of the product of matrices equals the product of their transposes in the reverse order.
- Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible and
$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$
  If $ad - bc = 0$ then A is not invertible.
- If A is an invertible $n \times n$ matrix, then for each b in $\mathbb{R}^n$, the equation $A\vec{x} = \vec{b}$ has the unique solution $\vec{x} = A^{-1}\vec{b}$.
- (1) If A is an invertible matrix then $(A^1)^{-1} = A$.
  (2) If A and B are $n \times n$ invertible matrices so is AB, and the inverse of AB is the product of the inverses of A and B in the reverse order. That is
$$(AB)^{-1} = A^{-1}B^{-1}.$$

(3) If A is an invertible matrix, then so is A^t, and the inverse of A^T is the transpose of A^{-1}. That is,

$$(A^T)^{-1} = (A^{-1})^T.$$

- An elementary matrix is one obtained by performing a single row operation on the identity matrix.
- If an elementary row operation is performed on an $n \times n$ matrix A, the resulting matrix can be written as EA, where the matrix E is created by performing the same row operation on I_n.
- Each elementary matrix E is invertible. The inverse of E is the elementary matrix of the same type that transforms E back into A.
- An $n \times n$ matrix A is invertible if and only if A is row equivalent to I_n, and in this case, any sequence of elementary row operations that reduces A to I_n also transforms I_n to A^{-1}. This justifies the validity of the Gauss-Jordan method for finding inverses.
- **The Invertible Matrix Theorem:** Let A be a square $n \times n$ matrix. The following statements are equivalent:
  (1) A is an invertible matrix.
  (2) A is row equivalent to the $n \times n$ identity matrix.
  (3) A has n pivot positions.
  (4) The equation $A\vec{x} = \vec{0}$ has only the trivial solution.
  (5) The columns of A form a linearly independent set.
  (6) The equation $A\vec{x} = \vec{b}$ has at least one solution for every $\vec{b}$ in $\mathbb{R}^n$.
  (7) The columns of A span $\mathbb{R}^n$.
  (8) There is an $n \times n$ matrix C such that $CA = I$.
  (9) There is an $n \times n$ matrix D such that $AD = I$.
  (10) A^T is an invertible matrix.
- Let A and B be square matrices. If $AB = I$, then A and B are both invertible with $A^{-1} = B$ and $B^{-1} = A$.
- **The Leontief Input-Output Model:**

$$\begin{array}{ccccc} \textit{(amount produced)} & = & \textit{(intermediate demand)} & + & \textit{(Final Demand)} \\ x & = & Cx & + & d \end{array}.$$

- Let C be the consumption matrix for an economy, and let $\vec{d}$ be the final demand. If C and $\vec{d}$ have nonnegative entries (i.e., all entries are ≥ 0, and if each column of C is less than 1, then $(I - C)^{-1}$ exists and the production vector

$$\vec{x} = (I - C)^{-1}\vec{d}$$

has nonnegative entries and is the unique solution of

$$\vec{x} = C\vec{x} + \vec{d}.$$