MATH 234: HOMEWORK 4

SOLUTIONS

1. A health club offers memberships at a rate of \$300 per person, provided that at least 50 people join. For each member in excess of 50, the membership fee will be reduced by \$2. So for example, if 51 people join the club, the membership will be \$298 per person. Due to space limitations, at most 125 memberships will be sold. How many memberships should the club sell in order to maximize its revenue?

Solution. Well there are several ways to go about this, since revenue is price times demand, let's try to write out a price function $P(x)$ which we will write in terms of demand x:

$$P(x) = 300 - 2(x - 50)$$

So revenue $R(x)$ is

$$R(x) = (300 - 2(x - 50))x = (300 - 2x + 100)x = 400x - 2x^2$$

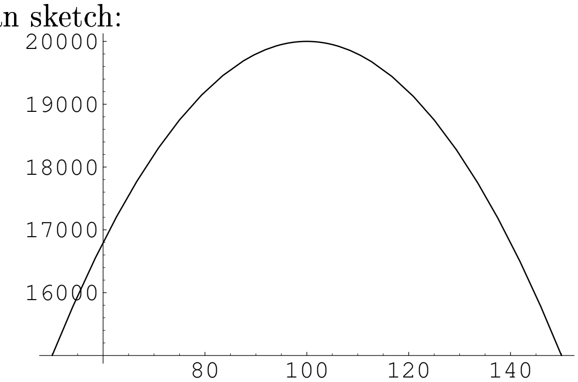
Now since we want to maximize revenue we'll sketch the curve. Look at

$$\frac{d}{dx}R(x) = 400 - 4x$$

and this is zero when $x = 100$. Is this truly a maximum? Well let's check the second derivative,

$$R''(x) = -4.$$

So now we can sketch:



Since this is negative and $R(x)$ is a quadratic, $x = 100$ maximizes the revenue.

2. In planning a sidewalk cafe, it is estimated that if there are 28 tables, the daily profit will be \$8 per table and that, if the number of tables is increased by x, the profit per table will be reduced by $x/4$ dollars (due to overcrowding). How many tables should be present in order to maximize the profit?

Solution. So we know profit is revenue minus cost. Let x be the number of tables added to 28 and $P(x)$ be the profit per table. So

$$P(x) = 8 - x/4,$$

while the number of tables is $28 + x$. So the total profit $T(x)$ is

$$T(x) = (8 - x/4)(28 + x) = 224 + x - x^2/4.$$

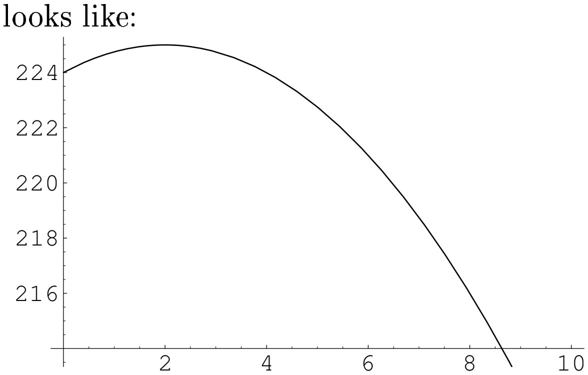
Since we want to maximize this, let's sketch the curve. Write

$$T'(x) = 1 - x/2.$$

Now $T'(x) = 0$ when $x = 2$. Is this truly a maximum? Well let's check the second derivative:

$$T''(x) = -1/2.$$

So our curve looks like:



Hence near $x = 2$ all the values are lower, and since $T(x)$ is a quadratic equation, this is a maximum. Hence one should set out 30 tables.

3. Compute the derivative with respect to r of $r^2(r-1)(r+1)^{-1}$.

Solution. Let's use the product rule!

$$\begin{aligned} &\frac{d}{dr} r^2(r-1)(r+1)^{-1} \\ &\quad = 2r(r-1)(r+1)^{-1} + r^2(r+1)^{-1} + r^2(r-1)(-1)(r+1)^{-2}. \end{aligned}$$

4. Compute the derivative with respect to x of $\dfrac{x+1}{x^2-1}$.

Solution. First do some algebra to make life easier:

$$\frac{x+1}{x^2-1} = \frac{x+1}{(x+1)(x-1)} = \frac{1}{x-1}.$$

Now I suppose we could use the quotient rule.

$$\frac{d}{dx}\frac{1}{x-1} = \frac{-1}{(x-1)^2}.$$

5. Compute the derivative with respect to x of $\dfrac{2x-7}{3x-2}$.

Solution. We can use the quotient rule.

$$\frac{d}{dx}\frac{2x-7}{3x-2} = \frac{(3x-2)2-(2x-7)3}{(3x-2)^2}.$$

6. Compute the derivative with respect to x of $(x^3+1)(3x^2-1)$.

Solution. Let's use the product rule!

$$\frac{d}{dx}\big((x^3+1)(3x^2-1)\big) = (3x^2)(3x^2-1)+(x^3+1)(6x).$$

7. Let $f(x) = x^3$. Using the chain rule, compute

$$\frac{d}{dx}f(g(x)).$$

Solution. Chain rule baby!

$$\frac{d}{dx}f(g(x)) = f'(g(x))\cdot g'(x),$$

and $f(x) = x^3$ so $f'(x) = 3x^2$, hence

$$\frac{d}{dx}f(g(x)) = 3(g(x)^2)\cdot g'(x).$$

8. Let $g(x) = \sqrt{x}$. Using the chain rule, compute

$$\frac{d}{dx}f(g(x)).$$

Solution. Well, $g'(x) = (1/2)(x^{-1/2})$. Now by the chain rule,

$$\frac{d}{dx}f(g(x)) = f'(\sqrt{x})\cdot(1/2)(x^{-1/2}).$$

9. Let $f(x) = x^2-9$ and $g(x) = x^2-16$. Compute

$$\frac{d}{dx}g(f(x)).$$

Solution. Ok by the chain rule,

$$\frac{d}{dx}g(f(x)) = g'(f(x)) \cdot f'(x).$$

But we know $f'(x) = 2x$ and $g'(x) = 2x$. Now we see

$$\frac{d}{dx}g(f(x)) = 4x(x^2 - 9).$$

10. Let $f(x) = 3/x + x^3$ and $g(x) = 1 - x^2$. Compute

$$\frac{d}{dx}f(g(x)).$$

Solution. We need to use our friend, the chain rule. So $f'(x) = -3x^{-2} + 3x^2$ and $g'(x) = -2x$. Now

$$\frac{d}{dx}f(g(x)) = (-3(g(x))^{-2} + 3(g(x))^2)g'(x)$$

But now write in $g(x)$ and $g'(x)$ to get

$$\frac{d}{dx}f(g(x)) = (-3(1-x^2)^{-2} + 3(1-x^2)^2)(-2x).$$

11. Suppose that x and y are related by the equation $x^3+(2y+1)^2 = y^2$. Use implicit differentiation to determine $\frac{dy}{dx}$.

Solution. Let's do it, write

$$3x^2 + 2(2y+1)(2)\frac{dy}{dx} = 2y\frac{dy}{dx}.$$

Now solving for $\frac{dy}{dx}$ we find

$$\frac{dy}{dx} = \frac{3x^2}{2y - 4(2y+1)}.$$

12. Assume $\frac{4}{x} + \sqrt{y} = x$. What is the slope of the graph at the point $(-1, 9)$?

Solution. We'll use implicit differentiation to solve this problem. Write

$$\frac{-4}{x^2} + (1/2)y^{-1/2}\frac{dy}{dx} = 1.$$

Solving for $\frac{dy}{dx}$ we find

$$\begin{aligned}\frac{dy}{dx} &= 2y^{1/2}(1+\frac{4}{x^2})\\ &= \sqrt{y}(2+\frac{8}{x^2}).\end{aligned}$$

Evaluating this at $x = -1$ and $y = 9$ we find

$$\frac{dy}{dx} = 30.$$