2.1 Row Operations and Gaussian Elimination

Suggested exercises: 1-33 odd
In chapter 1, we limited ourselves to linear equations with only two variables so that we could graph them. This led to situations where we only had two products or two types of food in a diet problem. In reality, we often have several products to manufacture, and we simply cannot graph these equations with several variables. We can, however, use algebra to solve the same types of problems as before.

In this section we will look at a way of mathematically computing with several equations and several variables. The benefit of this new notation is that we do not have to worry about things that don't matter, like variable names.

5 keys:

- Elementary operations
- Matrix of a system
- Elementary row operations
- Gaussian elimination
- Row echelon form

I Elementary operations

When we solved equations in two variables, we often used the method of elimination to find the points that solved both equations. We can think of the method of elimination as being made up of three basic operations, and each time we do one of these operations, we don't change our solution.

1. **Interchanging two equations.** If we write the equations in a different order, we certainly don't change any of the solutions.
2. **Multiplying an equation by a non-zero constant.** This just scales the equation, but doesn't change any of the solutions. (Be sure to multiply the number on the other side of the equals also!)
3. **Adding a multiple of one equation to another equation.** Forming the basis of the method of elimination, this is probably the most useful operation we have when we try to solve systems of linear equations.

Property 1 If a system of linear equations is derived from another system by applying any of the three elementary operations, then the solution sets of the two systems are identical.

Example.

Example.

II Matrix of a system

When we solve a system of equations, it doesn't really matter what we call the variables – as long as we line up our equations so the variables appear in columns, we can actually eliminate the variable names themselves. This gives us something called a *matrix*.

Definition A **matrix** is a rectangular array of numbers enclosed in brackets. The coefficients of a system of linear equations form a matrix called the **coefficient matrix** of the system. If we include a column for the constant terms on the right hand side of the equals signs, we call this matrix the **augmented (coefficient) matrix** of the system.

In an augmented coefficient matrix, each row corresponds to an equation in the system, and each column corresponds to a particular variable. The only exception is the last column, which represents the constants.

Example.

Example.

III Elementary row operations

Each of the elementary operations we discussed earlier can be done with the rows of a matrix.

1. Interchanging two rows of a matrix.
2. Multiplying a row by a constant.
3. Adding a multiple of one row to another.

These have the same effect on the original equations, but since we are now only working with the coefficients, it is easier to do the calculations.

Example.

IV Gaussian elimination

When we perform the method of elimination with a matrix, using the three elementary row operations, we are eliminating the variables from our original system as before. We call this procedure **Gaussian elimination**.

To perform Gaussian elimination:

1. Find the leftmost variable column that has a nonzero entry in it. Choose a nonzero entry in this column and interchange rows, if necessary, to put this entry in the top row.
2. Divide the top row by the nonzero entry found in Step 1 so that the leftmost entry in the row is 1. This entry is called the **leading 1** for the row.
3. Add or subtract multiples of the top row to the rows beneath it to obtain zeros beneath the leading 1.
4. Cover the top row and repeat Steps 1-3. Stop when all rows have been covered or when only entries with zeros remain in the uncovered rows.

Example.

After we do Gaussian elimination, we can turn our matrix back into equations and easily obtain our solution. The bottom nonzero row of our matrix corresponds to an equation with only one variable, and we can substitute this value back into previous rows. Repeating this procedure, we obtain our solution. This method is called **back substitution**.

Example.

V Row echelon form

The form that our matrix takes at the end of Gaussian elimination has some nice properties. In particular, the lower-left portion of the matrix is made up of zeros, forming something like a staircase. This special form is useful enough to have its own name.

Definition A matrix is said to be in **row echelon form** if

(i) the leftmost nonzero entry in each row is a 1 (its leading 1),

(ii) the entries below any leading 1 are 0,

(iii) the leading 1 for each row is to the left of the leading 1 for any row below it, and

(iv) any row of all zeros is located below the rows that have leading ones.

Example.

Example.