2.3 Consistency of Systems

Suggested exercises: 1-17 odd, 21-35 odd
All of our previous examples of systems of equations with three or more variables had exactly one solution. As we saw with two variables, this isn't always the case. In this section we examine the two other situations: no solutions and infinitely many solutions.

4 keys:

- Inconsistent systems
- Row rank
- Parametrically represented solutions
- Application

I Inconsistent systems

Recall that a system of linear equations is called **inconsistent** if it has no solutions; otherwise it is called **consistent**. A consistent system can have exactly one or infinitely many solutions.

Example.

II Row rank

Our row echelon form can tell us whether our linear system is consistent. We will need to count the number of nonzero rows in our coefficient matrix and compare that with the number of nonzero rows in our augmented matrix.

Definition The **row rank** of a matrix is the number of nonzero rows in a row echelon form of the matrix.

Example.

Property 1 A linear system is consistent if and only if the row rank of the coefficient matrix is equal to the row rank of the augmented matrix.

Example.

III Parametrically represented solutions

Now that we know how to determine whether we have any solutions, how do we know when we have infinitely many? We can look for columns in our coefficient matrix that don't have a leading 1. The variable represented by this column can be anything.

Example.

A new variable introduced in the solution is called a **parameter** or a **free variable**. The variables that have leading 1s in their columns are called **basic variables**. When we look for our solutions, we want to express the basic variables in terms of the free variables.

When we have infinitely many solutions, we can use the following steps to find all of the solutions:

1. *Put the augmented matrix of the system into reduced echelon form.* We use Gaussian elimination for this. Note that this is the reduced echelon form, not just the row echelon form.
2. *Write the system of equations represented by the reduced echelon form.*
3. *Replace each free variable by a parameter.* Be sure to use a different parameter for each free variable, and don't use a variable that is already being used.
4. *Express each basic variable in terms of the parameters.* In other words, solve for the basic variables. Since you put the matrix in reduced echelon form, the only other variables will be parameters.

Example.

IV Application

Our applications so far have had exactly one solution. But in reality, it is not surprising to have several solutions among which we can choose. We need to remember our "common sense" restrictions when we do this, however.

Example. A detergent company makes four laundry products: Surf, Cyclone, Breeze, and Brisk from three materials we will call AS, SP, and SS. The products are made in large batches, and the batches differ in size and proportions of raw materials because of different manufacturing processes and equipment. The amounts of the raw materials used to make 1 batch of each product and the total amounts on hand are given by the following table (in 100 lb. units):

	Surf	Cyclone	Breeze	Brisk	On hand
AS	4	8	4	4	60
SP	2	5	2	3	36
SS	3	7	4	3	50

Find the possible number of batches of each product that can be made such that all of the available raw materials are used.