

## 3.1 Matrix Operations

**Suggested exercises: 1-49 odd** (you will want to do many of these to get practice with matrices)

Matrices can be used for much more than just solving systems of linear equations. In this section we will look at some properties of matrices and see how they act like numbers most of the time. We will see an important exception to this, namely that multiplying matrices is not commutative.

**5 keys:**

- Matrices
- Matrix addition
- Scalar multiplication
- Matrix multiplication
- Properties of matrix multiplication

### I Matrices

**Definition** A **matrix** is a rectangular array of numbers. If a matrix  $A$  has  $m$  rows and  $n$  columns, we say that it is an  $m \times n$  matrix and call  $m$  and  $n$  the dimensions of  $A$ . The number of rows is always given first.

**Example.**

To refer to the entries of a matrix, we will use a lowercase letter with subscripts to tell which entry we are using. Again the row number will be listed first.

**Example.**

Matrices can represent different kinds of data, and whether the data is in a row or a column can be important. If we need to change the data from a row to a column or a column to a row, we find the *transpose* of the matrix.

**Definition** The **transpose** of a matrix is formed by taking each row and making it a column. In other words, if  $a_{ij}$  is the entry from row  $i$  and column  $j$ , then in the transpose matrix this entry is in row  $j$  and column  $i$ . We denote the transpose of a matrix  $A$  by  $A^t$ . If  $A$  is an  $m \times n$  matrix, then  $A^t$  will be an  $n \times m$  matrix.

**Example.**

## II Matrix addition

As long as two matrices  $A$  and  $B$  have the exact same dimension, we may add them to form  $A + B$ . We do this by simply adding each corresponding value and placing the sum in the same position in the new matrix.

**Definition** If  $A = [a_{ij}]$  and  $B = [b_{ij}]$  are matrices with the same dimension, then the **matrix sum**

$$A + B = C$$

is given by  $C = [c_{ij}]$ , where  $c_{ij} = a_{ij} + b_{ij}$  for all  $i$  and  $j$ .

**Example.**

**Example.**

## III Scalar multiplication

We can multiply an entire matrix by a single number, called a *scalar*. This is called scalar multiplication.

**Definition** If  $k$  is a scalar (constant) and  $A = [a_{ij}]$  is a matrix, then the **scalar multiplication** of  $A$  and  $k$  is defined by

$$kA = [c_{ij}]$$

where  $c_{ij} = ka_{ij}$  for all  $i$  and  $j$ .

**Example.**

If a matrix has only zeros for its entries, it is known as the **zero matrix**. We usually don't specify what  $m$  and  $n$  are for the zero matrix, and instead use context to infer this information.

We form the negative of a matrix by multiplying it by  $-1$ . This allows us to subtract matrices in the obvious way:  $A - B = A + (-1)B$ .

**Property 1 (Properties of scalar multiplication and addition of matrices)** Let  $a$  and  $b$  be scalars, and let  $A$  and  $B$  be  $m \times n$  matrices. Then we have the following properties:

- (i)  $A + B = B + A$
- (ii)  $A + (B + C) = (A + B) + C$
- (iii)  $a(bA) = (ab)A$
- (iv)  $(a + b)A = aA + bA$
- (v)  $a(A + B) = aA + aB$
- (vi)  $A + 0 = A$ , and  $a0 = 0$

## IV Matrix multiplication

Matrix multiplication is perhaps the most difficult operation for a matrix. However, once you learn the definition and practice, it can be done very easily.

We will start with matrices consisting of only 1 row or 1 column.

**Example.**

**Example.**

The definition of matrix multiplication breaks each matrix down into rows and columns, and applies this procedure to each pair. The entry in row  $i$  and column  $j$  of the product  $AB$  is given by multiplying row  $i$  from  $A$  and column  $j$  from  $B$ .

**Definition** If  $A = [a_{ij}]$  is an  $m \times n$  matrix and  $B = [b_{ij}]$  is an  $n \times p$  matrix, then the **matrix product**  $AB$  is the  $m \times p$  matrix

$$AB = c_{ij}$$

where  $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$  for all  $i$  and  $j$ .

**Example.**

**Example.**

## V Properties of matrix multiplication

We can also state the properties of multiplying two matrices. An important note is that matrix multiplication is *not* commutative:  $AB \neq BA!$

**Property 2 (Properties of matrix multiplication)** Let  $a$  and  $b$  be scalars, and let  $A, B$ , and  $C$  be matrices. Assume that the matrix dimensions are all appropriate for the sums and the products given below. Then, we have

(i)  $A(BC) = (AB)C$

(ii)  $(A + B)C = AC + BC$

(iii)  $A(B + C) = AB + AC$

(iv)  $0A = 0, A0 = 0$

(v)  $(aA)(bB) = abAB$

**Example.**

**Example.**

Notice that when a matrix  $A$  is square (so it is an  $n \times n$  matrix), we can multiply it by itself. We can form the powers of this matrix:  $A^2 = AA, A^3 = A^2A$ , etc.

**Example.**